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With respect to an orthonormal basis in Euclidean space E^{n+1}, the $n+1$ unit points are the vertices of an equilateral n-simplex. This suggests the question, "Is an arbitrary n-simplex congruent to a simplex with vertices of the form $(0, 0, \cdots, 0, a_i, 0, \cdots, 0)$, $i=0, \cdots, n$?" The answer in general is "no," since it is evident that a triangle containing an obtuse angle is not so placeable in E^3. The answer for $n \geqq 3$ is unexpected. An n-simplex is congruent to a simplex with vertices on the coordinate axes of E^{n+1} if and only if for each vertex, the pairs of edge vectors originating at the vertex have equal positive inner products. In case $n=2$, any acute angled triangle is placeable with vertices on the coordinate axes of E^3. The n vectors originating at a vertex, v_i, of an n-simplex, and terminating at the n adjacent vertices, will be referred to as "co-original edge vectors," e.g., ν_{ij} is the vector extending from vertex v_i to vertex v_j.

Investigation of this question is motivated in part by the following consideration. The placeability of a simplex on the coordinate axes is equivalent to the existence of a point, from which vectors to the vertices of the simplex are mutually orthogonal.

LEMMA 1. *A triangle may be placed with its vertices on the positive coordinate axes of E^3 if and only if the interior angles of the triangle are acute.*

Proof. Let $\phi_i=(\delta_{i0}, \delta_{i1}, \delta_{i2})$, $i=0, 1, 2$, be the standard orthonormal vectors along the rectangular coordinate axes. A triangle with vertices $a_0\ \phi_0$, $a_1\ \phi_1$, $a_2\ \phi_2$, $a_j \neq 0$, $j=0, 1, 2$, then has its lengths of sides a, b, c, given by $a^2=a_1^2+a_2^2$, $b^2=a_0^2+a_2^2$, $c^2=a_0^2+a_1^2$. The solutions for a_0^2, a_1^2, a_2^2 of the preceding equations are $a_0^2=(c^2+b^2-a^2)/2$, $a_1^2=(a^2+c^2-b^2)/2$, $a_2^2=(a^2+b^2-c^2)/2$. These are positive; thus the law of cosines gives the acute angle conclusion. Conversely, given any triangle in which the angles are acute, the preceding equations determine a_j, $j=0, 1, 2$, to place the triangle.

Lemma 2. *In the class of Euclidean n-simplices, the inner product values of pairs of co-original edge vectors at one vertex uniquely determine the inner product values at all of the other vertices. In particular, if they are constant at one vertex, they must be constant at each individual one of the other vertices, and at most one vertex can have an associated nonpositive value.*

Proof. Suppose that the $n+1$ vertices of the n-simplex have vector positions $v_j, 0 \leq j \leq n$ and that the inner product values of pairs of co-original edge vectors at v_0 are known. Without loss of generality, we assume that the vertex v_0 is positioned at the origin by a translation. This establishes $(\nu_{0i}, \nu_{0k}) = (v_i, v_k)$, $1 \leq i, k \leq n, i \neq k$, as known information. Consequently, the inner product values of edge vectors co-original at v_1 are given by:

$$\begin{aligned}(\nu_{1i}, \nu_{1k}) &= (v_i - v_1, v_k - v_1) \\ &= (v_1, v_1) - (v_1, v_k) - (v_i, v_1) + (v_i, v_k) \qquad 2 \leq i, k \leq n \quad i \neq k \\ (\nu_{10}, \nu_{1k}) &= (-v_1, v_k - v_1) \\ &= (v_1, v_1) - (v_1, v_k) \qquad 2 \leq k \leq n.\end{aligned}$$

If the values (v_i, v_j), $i \neq j$, are constant, say c, then the above equalities give:

$$\begin{aligned}(\nu_{1i}, \nu_{1k}) &= (v_1, v_1) - c \qquad 2 \leq i, k \leq n, i \neq k \\ (\nu_{10}, \nu_{1k}) &= (v_1, v_1) - c \qquad 2 \leq k \leq n.\end{aligned}$$

Hence a constant value, c, at the vertex v_j establishes a constant value, $(v_i, v_i) - c$, at each of the other vertices v_i, $i \neq j$. In case $c \leq 0$, then $(v_i, v_i) - c > 0$, $i \neq j$. Consequently at most one vertex can have a constant nonpositive value.

Theorem 1. *An n-dimensional simplex may be placed with its vertices on the coordinate axes of E^{n+1}, if and only if all the triangle faces have acute interior angles, and at one vertex pairs of co-original edge vectors give equal inner products.*

Proof. Suppose that the $(n+1)$ vertices of an n-simplex are $a_0\phi_0, a_1\phi_1, \cdots, a_n\phi_n$, where again $\phi_0, \cdots, \phi_n$ are the standard orthonormal vectors, and $a_j \neq 0, j = 0, \cdots, n$. Then the n edge vectors, at say vertex v_0, are $a_1\phi_1 - a_0\phi_0, \cdots, a_n\phi_n - a_0\phi_0$, and we see that the inner product of any pair of these has value a_0^2. Similarly, for vertex v_i, the inner product of any pair of edge vectors is a_i^2. Thus a placeable n-simplex has a positive inner product value a_i^2 associated with each vertex v_i $(i = 0, \cdots, n)$, and in particular all angles between co-original edge vectors must be acute.

By Lemma 2, an n-simplex that satisfies the given conditions has a positive inner product value a_i^2 associated with each vertex v_i $(i = 0, 1, \cdots, n)$. The given simplex is congruent to the n-simplex described by the points

$$u_i = (\delta_{0i}a_0, \delta_{1i}a_1, \cdots, \delta_{ni}a_n)$$

$(i = 0, 1, \cdots, n)$. This follows from the previous Lemma 1 when we note that we have congruent facial triangles and consequently congruent side lengths in the proper correspondence.

It is interesting to note the following corollary. The proof is immediate from calculations arrived at after the tetrahedron ($n=3$) is placed. A similar relationship holds for nonadjacent edges of a placeable n-simplex for any $n>3$, but the related nonadjacent edges must be formed with end points of the original pair.

COROLLARY. *If a tetrahedron is placeable on the positive coordinate axes of E^4, then the sum of the squares of the lengths of two nonadjacent edges is equal to the sum of the squares of the lengths of any other two nonadjacent edges.*

The necessity of the acute angle hypothesis is demonstrated by the following example. Consider an n-simplex with the origin of E^n as one vertex and the other n vertices expressed as $v_i=(-a, -a, \cdots, -a, 1, -a, \cdots, -a)$, with the 1 in the ith position. Here the edge vectors co-original at the origin have pairwise inner products $(n-2)a^2-2a$, which are negative when $0<a<2/(n-2)$. Likewise the edge vectors co-original at v_i yield equal pairwise inner products $(1+a)^2>0$. Thus, we have an illustration of a nonplaceable simplex satisfying the equal inner product condition. The same example restricted to a tetrahedron shows that the corollary does not express a sufficient condition for placeability.