CONVEX POLYGONS

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1. **Introduction and terminology.** In the two-dimensional Euclidean plane E_2, if γ, δ are any linearly independent vectors, define projections P, Q by $P(x\gamma+y\delta)=x\gamma$, $Q(x\gamma+y\delta)=y\delta$. Denote by L, M the lines consisting respectively of the set of points $\{a\gamma\}$ for all real a, and of the set of points $\{b\delta\}$ for all real b. A convex region K in E_2 is of *standard type, by projections parallel to the directions of* γ, δ, in case there is a translate C of K, and a choice of vectors γ, δ, such that the projections PC, QC of C coincide respectively with the sections $L\cap C$, $M\cap C$ of C with the lines L, M.

For example, any rectangle is of standard type by projections P, Q parallel to its diagonals; it is also of standard type by projections parallel to its sides. A triangle is of standard type by projections P, Q parallel to a side and to any chord joining the opposite vertex to a point of the side.

By saying that a region C is *symmetric*, we mean that it can be so translated in E_2 that it is symmetric or centered in the origin θ of E_2; that is, whenever $(x, y)=\sigma=x\gamma+y\delta$ is a point of C, so is $-\sigma=(-x, -y)$. In dealing with a symmetric region, we always assume that it is so located in E_2 that its center is at θ.

We are now able to state the principal results of this paper. *Every closed symmetric convex polygon C is of standard type, both by projections parallel to a pair of diagonals of C, and by projections parallel to a pair of sides of C.* Each nonsymmetric closed convex polygon D has an associated symmetric polygon C, with sides and diagonals parallel to those of D. Therefore by the results for symmetric polygons, D also is of standard type, by projections parallel to a pair of sides, and by projections parallel to a pair of diagonals. By polygonal approximations, this implies that *every bounded planar convex region is of standard type.* In the final section, various functional analytic implications are derived, and open questions and a conjecture are indicated.

2. **Diagonal projections of symmetric convex polygons.** For any symmetric closed convex polygon C_p having $2p$ sides, denote by $\alpha_1, \cdots, \alpha_p$ the angles, measured from a horizontal ray from θ, of the

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sides $1, \cdots, p$ in counterclockwise succession, and by $\beta_0, \cdots, \beta_{p-1}$, the angles of the diagonal radii from θ to the initial points of sides $1, \cdots, p$. A rectangle is of standard type by projections parallel to its diagonals; we shall now prove by induction that C_p is of standard type by projections parallel to a pair of its diagonals.

Let $V_1, \cdots, V_p$ denote the vertices (terminal points of sides $1, \cdots, p$) of C_p. Then vectors to the opposite vertices $-V_1, \cdots, -V_p$ of C_p have respective angles $\beta_1 \pm \pi, \cdots, \beta_{p-1} \pm \pi, \beta_0$. If the pair of sides $i, i+1$ and the opposite pair of sides are replaced by single sides joining V_{i-1} to V_{i+1} and $-V_{i-1}$ to $-V_{i+1}$, we obtain a symmetric convex polygon $C_{p-1,i}$ which has $2(p-1)$ sides. The induction hypothesis is that every C_{p-1} if of standard type by projections parallel to a pair of diagonals of C_{p-1}.

If the projections by which $C_{p-1,i}$ is of standard type are in directions β_h, $h < i-1$, and β_j, $j > i+1$, then in consequence of convexity of C_p, C_p is of standard type by the same projections. In case $h = i-1$, C_p is of standard type by the same projections provided that $\beta_j \leqq \alpha_i$; and in case $j = i+1$, C_p is of standard type by the projections provided that $\alpha_{i+1} \leqq \beta_h + \pi$. Either or both inequalities are satisfied, if they need to be for the induction, at a vertex V_i such that $\alpha_{i+1} \leqq \beta_{i-1} + \pi$ and $\beta_{i+1} \leqq \alpha_i$, where $\alpha_{p+1} = \alpha_1 + \pi$, $\beta_p = \beta_0 + \pi$, $\beta_{p+1} = \beta_1 + \pi$.

In case C_p is such that there is no vertex V_i at which both inequalities of the above pair are satisfied, so that we cannot conclude that C_p is of standard type by the projections for some $C_{p-1,i}$, then for each $i = 1, \cdots, p$, we have that either

$$\beta_{i+1} > \alpha_i \tag{1}$$

or

$$\alpha_{i+1} > \beta_{i-1} + \pi \tag{2}$$

or both, are satisfied. For any i, by convexity $\alpha_{i+1} > \alpha_i$, and $\beta_{i-1} + \pi > \alpha_i > \beta_i$. Consider, for example, vertices V_1 and V_2. In case say $\beta_2 > \alpha_1$ and $\alpha_3 > \beta_1 + \pi$, then also $\beta_1 + \pi > \alpha_2 > \beta_2$, so in this case C_p is of standard type by projections in directions β_1, β_2.

For any possible C_p as in the preceding paragraph, we may indicate which of the inequalities (1), (2) is satisfied at each vertex $V_1, \cdots, V_p$, by a binary symbol having p digits, such as $121 \cdots 1$. Digit 1 indicates that (1) is satisfied, digit 2, that (2) is satisfied. (If both are satisfied at V_k, the choice between digits 1 and 2 for the kth place is arbitrary.) If the binary symbol for a C_p contains anywhere a successive pair of digits 12, or if it begins with 2 and ends with 1, then C_p is similar to the example of the preceding paragraph, and therefore is of standard type by projections parallel to a successive pair of diagonals.

The only remaining cases are those for which the binary symbol consists of either all 1's, or all 2's. We now complete the proof by showing that these cases for C_p are impossible. The case of all 2's becomes the case of all 1's if the numeration and positive sense of angles are changed from counterclockwise to clockwise (or if the plane of C_p is inverted), so it is sufficient to dispose of the case of all 1's. We therefore assume that $\beta_{i+1}>\alpha_i$, $i=1, \cdots, p$. By convexity, there is a vertex V_k such that a line at angle β_0 through V_k is a support line of C_p. By affine transformation, we may suppose that the difference $\beta_k-\beta_0$ is $\pi/2$, and that $\beta_0=0$. In case $1<k\leqq p-1$, if v_1 is the vertical projection of side 1, we have $d_0<v_1 \cot \beta_1$, where d_0 is the length of the horizontal diagonal chord. If d_k is the length of the vertical diagonal chord, by hypothesis we have $d_k>v_1$, and $d_0>d_k \cot(\alpha_p-\pi)$. But since $\beta_{p+1}=\beta_1+\pi>\alpha_p$, we have $\cot \beta_1<\cot(\alpha_p-\pi)$, $d_0>v_1 \cot(\alpha_p-\pi)$ $>v_1 \cot \beta_1$, which contradicts $d_0<v_1 \cot \beta_1$. If $k=1$ and $p>3$, then $\alpha_2\geqq\pi$, $\beta_3>\alpha_2$, but since $\beta_0=0$, we have $\beta_3<\pi$, which is a contradiction. If $k=1$ and $p=3$, then by hypothesis $\alpha_2\geqq\pi$, and also $\beta_3=\beta_0+\pi$ $=\pi>\alpha_2$, which again is a contradiction. Therefore the case of all 1's is impossible, and it has been demonstrated fully that every C_p is of standard type by projections parallel to some pair of its diagonals.

3. **Duality of projections parallel to sides and diagonals.** Points and lines in E_2 are related to lines and points in the dual plane E_2^* of E_2 by means of the basic equation

$$x\xi + y\eta - 1 = 0. \tag{3}$$

A point (x, y) of E_2 is dual to the line of points whose coordinates (ξ, η) in E_2^* satisfy equation (3); the line of E_2 given by (3) for fixed (ξ, η) is dual to the point (ξ, η) in E_2^*. As usual the finite planes E_2, E_2^* may be enlarged to be projective planes by adjunction of ideal points at infinity. Then θ is dual to the line at infinity of E_2^*; each line $x \cos \omega+y \sin \omega=0$ through θ in E_2 is dual to the point at infinity in the direction ω of E_2^*.

The dual points $U_1, \cdots, U_p$, and $-U_1, \cdots, -U_p$, of the sides $1, \cdots, p$, and of the parallel opposite sides, are vertices of the dual polygon C_p^* of C_p. It is easily shown that C_p^* is symmetric and convex. If C_p is of standard type by projections parallel to its diagonals at angles β_h, β_j, denote by B_h, B_j the lines containing the corresponding diagonals, and by B_h', B_j' the parallel lines respectively through vertices V_j, V_h. The dual point $B_h'^*$ is a point of side U_jU_{j+1} of C_p^*;

since B_h passes through θ, the dual point B_h^* is a point at infinity. Since B_h and sides h, $h+1$ are copunctal, the points B_h^*, U_h, U_{h+1} are collinear; that is, the line through $B_h'^*$ in the direction of B_h^* is parallel to side U_hU_{h+1}. Similarly the line through the point $B_j'^*$ on side U_hU_{h+1} in the direction of the point at infinity B_j^* is parallel to side U_jU_{j+1}. Therefore C_p^* is of standard type by projections parallel to sides U_hU_{h+1} and U_jU_{j+1}.

Any closed symmetric convex polygon may be taken as a C_p^*; then since $C_p^{**}=C_p$ is of standard type by projections parallel to a pair of diagonals, C_p^* is of standard type by projections parallel to a pair of sides. Direct proof that every C_p is of standard type by projections parallel to a pair of sides seems to be more difficult; or at least the author, after some effort, abandoned his attempt to find a direct proof, and succeeded in finding the preceding proof by appeal to duality.

4. Nonsymmetric convex polygons. If D is any closed convex polygon having n sides, a direction of angular reference may be chosen so that if $\alpha_1, \cdots, \alpha_n$ are the angles of the sides of D, then $0<\alpha_1< \cdots <\alpha_n<2\pi$, where the sides are taken in counterclockwise succession. Let $A_1, \cdots, A_n$ be vectors for the sides of D, in corresponding counterclockwise directions around D. A *diameter* of D is a chord which joins the points of contact of parallel support lines of D.

We may associate with D a symmetric convex polygon C as follows: $C=D-D$ is the set of all vectors of the form $(X-Y)$, where $X\in D$, $Y\in D$, with respect to an arbitrary origin θ. (P. C. Hammer has named the symmetric convex body C so associated with any convex body D the *symmetroid* of D; the concept also occurs in work of V. L. Klee.[2])

It may be easily shown that the vectors for the sides of C are $\pm A_1, \cdots, \pm A_n$, where the angle of $-A_k$ is $(\alpha_k-\pi)$ if $\alpha_k \geqq \pi$, $(\alpha_k+\pi)$ if $\alpha_k<\pi$, and the sides $\pm A_1, \cdots, \pm A_n$ are arranged in order of increasing angles of the vectors. Diagonals of D which are diameters correspond to diagonal chords of C; diameters from a vertex to a side of D correspond to diametral chords between a pair of parallel sides of C. Therefore obviously since C is of standard type by projec-

[2] It also appears earlier in work of O. B. Ader [1, p. 297], and implicitly in the treatise of Bonnesen and Fenchel [2]. The "support-function" $H(\xi, \eta)$ for a convex region D is defined by the equation for the support-line in any direction (ξ, η), $x\xi+y\eta-H(\xi, \eta)=0$. The function $H(\xi, \eta)$ is the "gauge-function" for the dual region D^*. The associated symmetric convex region C is defined by Bonnesen and Fenchel to be the region which has the "support-function" $H(\xi, \eta)+H(-\xi, -\eta)$; it may be easily verified that $C=D-D$.

tions parallel to a pair of (nonparallel) sides, D is of standard type by projections parallel to the corresponding sides of D; similarly D is of standard type by projections parallel to some pair of diametral diagonals of D.

Let a side A_k of D such that $(\alpha_{k+1}-\alpha_{k-1}) \geqq \pi$, if any, be called an *obtuse* side. Then clearly if D has an obtuse side, it is like a triangle, and is of standard type by projections parallel to the side and to any ray at an angle β such that $\alpha_{k-1} \leqq \beta \leqq \alpha_{k+1}-\pi$. All sides of a triangle are obtuse; any convex skew quadrilateral must have an obtuse side; a convex pentagon may have all sides nonobtuse. Since a convex polygon D having no obtuse side is of standard type by projections parallel to a pair of interior diagonals, for such a D there always exists a diametral diagonal which is an obtuse side for two polygons into which D is divided by the diagonal; the two polygons are simultaneously of standard type by projections parallel to the diagonal and to the segments into which the other diagonal is divided.

For any vertex V_i of a convex polygon D, consider particular chords from V_i as follows. In case D has an odd number of sides, the chords (from V_i to an interior point of a side of D) are those which divide D into two polygons with the same number of sides. In case D has an even number of sides, the chord is the diagonal (from V_i to another vertex of D) which divides D into two polygons with the same number of sides. Iff such a chord is a diameter of D, then the two polygons into which D is divided are each of standard type by projections parallel to the chord and to the support lines of the end points of the chord (although D itself need not be of standard type by those projections). Call a vertex V_i, such that the particular chords or chord as described are diameters of D, a *regular* vertex. Iff V_i is regular, the triangle whose sides are extensions of the sides of D adjacent to V_i, and whose base is in the support line at the other end of the diameter from V_i, circumscribes D.

THEOREM. *Every convex polygon D has at least one regular vertex.*

PROOF. Suppose first that the number n of sides of D is odd. By definition, for $j>k$, V_j is regular in case $(\alpha_j-\alpha_{j-k})<\pi$, and $(\alpha_{j+1}-\alpha_{j-k}) > \pi$, where $\alpha_{n+1} = (\alpha_1 + 2\pi)$. For $j \leqq k$, V_j is regular in case $\alpha_j-(\alpha_{j+k+1}-2\pi)<\pi$, and $\alpha_{j+1}-(\alpha_{j+k+1}-2\pi)>\pi$. Thus assuming that every vertex is not regular, we have

$$(4) \qquad (\alpha_j - \alpha_{j-k}) \geqq \pi, \quad \text{or} \quad (\alpha_{j+1} - \alpha_{j-k}) \leqq \pi,$$

for $j>k$, and

(5) $(\alpha_{j+k+1} - \alpha_j) \leqq \pi$, or $(\alpha_{j+k+1} - \alpha_{j+1}) \geqq \pi$,

for $j \leqq k$. For each j, by convexity and since D has n sides, not both inequalities in (4) or in (5) can be satisfied. Assume that the first member of (4) is true for $j=k+1$; the second member, which is the same as the first member of (5) for $j=1$, then is false. The second member of (5) for $j=1$, which is the same as the first member of (4) for $j=k+2$, then is true. The second member of (4) for $j=k+2$, which is the same as the first member of (5) for $j=2$, then is false. Continuing in this way, we see that on the basis of the assumption that the first member of (4) is true, all the first members of (5) are false, and all the first members of (4) are true. In particular for $j=2k+1=n$, $(\alpha_n - \alpha_{n-k}) \geqq \pi$, and so the second member of (4), namely $\alpha_{n+1} - \alpha_{k+1} \leqq \pi$, is false; that is $(\alpha_1 + 2\pi) - \alpha_{k+1} > \pi$, or $\alpha_{k+1} - \alpha_1 < \pi$. But this contradicts the assumption that the first member of (4) for $j = k+1$, namely $\alpha_{k+1} - \alpha_1 \geqq \pi$, is true. Similarly if we assume that the first member of (4) for $j=k+1$ is false, from the second member of (4) for $j=n$ and the intervening inequalities, we obtain a contradiction. Therefore, we have now established that D must always have at least one regular vertex.

For the case of even n, $n=2k$, a vertex V_j is regular iff $(\alpha_j - \alpha_{j-k}) \leqq \pi$ and $(\alpha_{j+1} - \alpha_{j-k}) > \pi$; that is, in case there is a triangle T circumscribing D, such that V_j and extended sides j, $j+1$ are a vertex and adjacent sides of T, and the opposite vertex V_{j-k} is on the third side of T. Consideration of the similar system of alternative inequalities to (4) and (5) yields the conclusion that D has at least one regular vertex.

REMARKS. In case no side of D is parallel to a diametral diagonal, then D is of standard type *only* by projections parallel to sides, or by projections parallel to diagonals. If B_h, B_j are the diagonals such that D is of standard type by projections parallel to B_h, B_j, and if the origin θ in E_2 is chosen at the point of intersection of B_h and B_j, then the dual polygon D^* is a convex polygon which has two pairs of parallel sides, and D^* is of standard type by projections parallel to the directions of the pairs of parallel sides. Any convex polygon, which has two pairs of parallel sides, is of standard type by projections parallel to the two directions of the pairs of parallel sides, if and only if there is a diameter parallel to each of the two directions, which joins points of the parallel sides in the other direction. This is a dual condition for D to be of standard type by projections parallel to a particular pair of diametral diagonals.

5. **Functional analytic interpretations and equivalent properties.** Any bounded symmetric convex region C in E_2 may be taken as the unit cell for a Minkowski or Banach norm on E_2. It follows from the result of §2 that basis vectors γ, δ may always be chosen so that this norm simultaneously has the properties $\|(x+\Delta x)\gamma+y\delta\| \geqq \|x\gamma+y\delta\|$ when $x, \Delta x$ have the same sign, $\|x\gamma+(y+\Delta y)\delta\| \geqq \|x\gamma+y\delta\|$ when $y, \Delta y$ have the same sign, and $\|x\gamma+y\delta\| \geqq \max(\|x\gamma\|, \|y\delta\|)$. Another equivalent property is that C is intermediate between the convex hull and the Cartesian product of the intervals $L\cap C$, $M\cap C$. (The norm $\|x\gamma+y\delta\|_1=\|x\gamma\|+\|y\delta\|$ has the convex hull for its unit cell.)

Still another equivalent statement is the following: if γ, δ are variable vectors of unit norm, and σ, τ are unit tangent or support vectors to the unit cell C at the end points of γ, δ, in directions such that the couples of vectors (γ, σ) and (δ, τ) have opposite senses, then there always exist γ, δ such that $\tau=\gamma$ and $\sigma=\delta$. (The author was unable to establish this directly by application of the Brouwer or other fixed point theorem.) Since a support line exists at the end point of every γ, we may take $\delta=\sigma$, and assert that a γ always exists such that $\tau=\gamma$.

It is a theorem of Banach and Mazur [**3**, pp. 185–187], that every separable Banach space is isometric with a subspace of the space (C) of continuous functions $z(t)$ on a finite closed interval, where the norm in (C) is given by $\|z\|=\max_t |z(t)|$. In particular, the norm for any two-dimensional Banach space L_2 is given by $\|x\gamma+y\delta\| =\max_t|x\gamma(t)+y\delta(t)|$, where $\gamma(t)$ and $\delta(t)$ are any pair of continuous functions which span the subspace M_2 of (C) which is isometric with L_2.

In case of a Banach space L_2 such that the unit cell is a symmetric convex polygon C of $2p$ sides, the functions γ, δ may be functions on a discrete set of p or $2p$ elements. If we change the independent variable from t to $u=\arctan \delta/\gamma$, if $x\xi_i+y\eta_i-1=0$, $i=1, \cdots, 2p$, are the equations of the sides of C, then we may set $\gamma(u_i)=\xi_i$, $\delta(u_i)=\eta_i$, where $u_i=\arctan \eta_i/\xi_i$, and γ, δ may be extended to be continuous functions on the entire interval $0\leqq u<2\pi$ by interpolating, for each i, the continuous graph between u_i and u_{i+1} which corresponds to the line segment joining the points (ξ_i, η_i) and (ξ_{i+1}, η_{i+1}) in the dual plane (the points of this line segment are dual to all the support lines through vertex V_i in the original plane).

For any unit cell C, we may approximate C by a sequence $\{C_n\}$ of circumscribing symmetric convex polygons, with the number of sides of C_n increasing indefinitely with n. Further, we may choose the circumscribing polygons so that from a certain stage on they are all of standard type by projections parallel to a pair of sides which maintain constant directions in the succession of circumscribing polygons.

Thus if $\gamma_n(u)$, $\delta_n(u)$ are the extended functions corresponding to one of the polygons C_n, if C_n is of standard type by projections parallel to sides h, k, then we have $\max_u |x\gamma_n(u)+y\delta_n(u)| = |x\gamma_n(u_h)+y\delta_n(u_h)| = |x\gamma_n(u_h)|$ for sufficiently small y, so $\delta_n(u_h)=0$, and similarly $\gamma_n(u_k)=0$. Therefore the functions $\gamma(u)$, $\delta(u)$ which represent C have the property that $\delta(u_1)=0$ at a value $u=u_1$, where $\gamma(u)$ is maximum, and simultaneously $\gamma(u_2)=0$ at a value $u=u_2$ where $\delta(u)$ is maximum. This, and similar considerations for any bounded convex D in E_2 with θ interior to D, yield the following theorem.

THEOREM. *If $\alpha(u)$, $\beta(u)$ are any pair of linearly independent continuous functions, periodic with period 2π, then there exist linear combinations $\gamma(u)=a\alpha(u)+b\beta(u)$, $\delta(u)=c\alpha(u)+d\beta(u)$, such that simultaneously $\delta(u_1)=0$ at a value $u=u_1$ where $\gamma(u)$ is maximum, and $\gamma(u_2)=0$ at a value $u=u_2$ where $\delta(u)$ is maximum. If in particular α, β are continuously differentiable, then there always exist distinct values u_1, u_2 such that simultaneously $\alpha'(u_1)\beta(u_2)=\alpha(u_2)\beta'(u_1)$ and $\alpha'(u_2)\beta(u_1)=\alpha(u_1)\beta'(u_2)$.*

This theorem follows since any polygon D_n is of standard type by projections parallel to a pair of sides, and since the function on E_2 defined by $p(x, y)=\max_u(x\alpha(u)+y\beta(u))$, for any pair of independent continuous functions α, β, is positive homogeneous and subadditive, which implies that $D=\{(x, y): p(x, y)\leqq 1\}$ is convex and has θ in its interior. The function defined by $\|(x, y)\|=\max_u |x\alpha(u)+y\beta(u)|$, where α, β are such that $\alpha(u+\pi)=-\alpha(u)$, $\beta(u+\pi)=-\beta(u)$, is a norm.

A convex region K in m-dimensional Euclidean space E_m is of *standard type* if there exist a translate D of K, and a choice of basis $\gamma_1, \cdots, \gamma_m$, such that $P_jD=L_j\cap D$, where $P_j(x_1\gamma_1+\cdots+x_m\gamma_m)=x_j\gamma_j$, and L_j is the line of points $\{c\gamma_j\}$ for all real c. The author ventures the following conjecture.

CONJECTURE. *Every bounded convex region D in E_m is of standard type.*

A form of the conjecture, which is weaker for nonsymmetric convex regions, would allow the ranges of the projections and lines L_j to be translates of the lines $\{c\gamma_j\}$. By the existence of the associated symmetric convex region, as indicated in §4, to establish the truth of at least the weaker form of this conjecture, it would be sufficient to prove it for bounded symmetric convex regions C in E_m. One method of attempting to prove it might be to apply the above theorem in some way to a set of m linearly independent continuous functions $\alpha_1(u), \cdots, \alpha_m(u)$, having the property $\alpha_j(u+\pi)=-\alpha_j(u)$, $j=1, \cdots, m$, which corresponds to symmetry of C. The unit cell for the norm $\|(x_1, \cdots, x_m)\| = \max_u |x_1\gamma_1(u) + \cdots + x_m\gamma_m(u)|$ is of standard type if there exist distinct values $u_1, \cdots, u_m$, such that $\gamma_j(u)$ is maximum at $u = u_j$, and $\gamma_i(u_j) = 0$, $i \neq j$, for $j = 1, \cdots, m$.

An unsolved problem concerning separable Banach spaces M is to show whether or not every such space has a base, that is, a set of elements $\{\gamma_j\}$ of M such that each element X of M has a unique expansion $X = \sum_{i=1}^{\infty} x_i\gamma_i$, which converges in norm to X. (See [3, pp. 110–111, and p. 238].) In case M has a base, the projections $P_jx = x_j\gamma_j$, although uniformly bounded, may have bounds greater than or equal to 2. If the unit cell C for a separable Banach space M is of standard type, then M has a base $\{\gamma_j\}$ with the stronger property that the projections P_j are all of norm 1.

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