CONVEXITY AND RELATED COMBINATORIAL GEOMETRY

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GRAPHICAL DIFFERENCE SETS AND PROJECTIVE PLANES*

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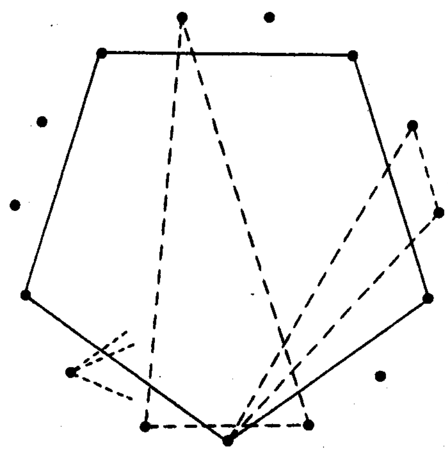
1. INTRODUCTION

One justification for presentation of a paper in combinatorics at a conference on convexity is of course the combinatorial structure of convex polytopes. In space R^d of dimension $d = 6k$ or $d = 6k + 2$, the skeleton-edge structure of a simplex is a Steiner system on $6k + 1$ or $6k + 3$ vertices; i.e., there are sets of edge-disjoint 2-faces or triangles which cover all of the edges, each exactly once. At the problem sessions of this conference, the author will pose questions like the following: Associate K_{15}, the complete graph on fifteen vertices, with a regular planar polygon (see Fig. 1). The edges are classified, by numbers of vertices "skipped," as 0, 1, 2, 3, 4, 5, $6 \equiv 7$. All skip-2 edges are covered by the three equiskip-2 5-circuits, and the remaining skips by two rotational series of fifteen triangles each, the skip 0, 3, 4 triangles and the skip 1, 5, 6 triangles. Is there a 13-polytope having fifteen vertices, which correspondingly has thirty triangular 2-faces and three pentagonal 2-faces, all edge-disjoint? (The edges of K_{15} are uniquely covered also by the three 5-circuits and by fifteen quadruples with outer skips 0, 3, 1, $7 \equiv 6$, inner skips 4, 5. In other terms, the

*Presented by title

FIGURE 1



nonskip-2 edges are covered by the fifteen skip 0, 3, 4 triangles, and the fifteen 1, 5, 6 "forks" or basic matroid elements of K_{15}.)

Here is another question of a similar nature: The edges of K_{39} are uniquely covered by two series of thirty-nine quadruples, those with respective exterior skips 0, 17, 8, 10; 1, 13, 16, 5, by thirteen equiskip-12 triangles, by the thirty-nine skip 2, 3, 6 triangles, and by the thirty-nine skip 4, 9, 14 triangles. Is there a polytope, in R_d of appropriate maximal possible dimension d, which correspondingly has seventy-eight rectangular 2-faces and ninety-one triangular 2-faces? Perhaps not all of the triples and quadruples can correspond to outer 2-faces; if so, this would reduce the maximum possible dimension d. Skips for the several series may be clockwise (c.) or counterclockwise (cc.); is it possible that there are polytopes for some combinations (c., cc., cc., c., etc.) but not for others?

Also, the author believes that a viable topic is convex sets in the vector spaces over the Galois fields GF(p) and GF(p^k) [p prime, k an integer > 1]. The points of the affine plane AG(2,p^k) are the ordered pairs (x,y), x, y $\in$ GF(p^k) [$k \geq 1$]. As usual, the lines have equations of the form $y = mx + b$, or $x = c$ in case of infinite slope, m, b, c $\in$ GF(p^k). Thus if $n = p^k$, AG(2,n) has n^2 points, and the $n^2 + n$ lines fall into $(n + 1)$ bundles of n parallel lines each. For each point P(x,y), there is a pencil of $(n + 1)$ lines which pass through P. As soon as the author establishes the existence of finite vector spaces R_d which are not the sets of d-tuples $(X_1,\ldots,X_d)$, $X_i \in GF(P^k)$ but which otherwise have the usual geometric properties of affine R_d (i.e., of finite affine spaces of orders different than p^k), the "convex-setters" (axiomatic and nonaxiomatic) may go to work in such spaces!

In view of results quoted, e.g., on pp. 27-28 of [2], with the usual axioms for affine and projective spaces, for $d \neq 2$ there are no affine or projective d-spaces other than Galois (= Desarguesian). An (unusual) affine d-space, $d \neq 2$, such as the author visualizes, might involve matroids [6], nearfields, ternary rings, neofields [5], or other generalizations of Galois fields.

To extend AG(2,n) to be the projective plane PG(2,n), as usual for each slope there is adjoined a point at infinity; the line at infinity consists of all the points at infinity. Thus each line of PG(2,n) has (n + 1) points, and the total number of points, as well as the total number of lines, becomes $(n^2 + n + 1)$. For n other than $n = p^k$, the existence of AG(2,n) is equivalent to the existence of a corresponding PG(2,n).

2. ROTATIONAL MODELS FOR PROJECTIVE PLANES

A line contains all of its segments; in that respect a line having (n+1) points is analogous to an n-simplex. The projective plane of order 2 may be viewed as the rotational Steiner system on seven vertices. The edges of K_7 are uniquely covered by the seven skip 0, 1, 2 triangles, which are the seven lines of PG(2,2). The six lines of AG(2,2) may be regarded as the six edges of K_4.

A rotational model for PG(2,3) is provided by K_{13} associated with a polygon having thirteen vertices, the lines being the thirteen quadruples which have outer skips 1, 0, 3, 5, inner skips 2, 4. The quadruples uniquely cover all edges of K_{13}.

There are similar rotational models for all the projective planes PG(2,n) over $GF(p^k)$ $[n = p^k]$, at least for n = 4, 5, 7, 8, 9, 11. For n = 4, the twenty-one lines are the twenty-one quintuples of K_{21} having outer skips 2, 0, 4, 1, 9, inner skips 5, 8, 6, 7, 3. For n = 5, the thirty-one lines (in K_{31}) are the thirty-one sextuples having outer skips 3, 5, 12, 0, 1, 4, inner skips 8, 9, 18 ≡ 11, 13, 2, 6, 7, 10, 15 ≡ 14. For n = 7, the outer skips are 4, 0, 1, 9, 18, 3, 6, 8, and inner skips the remaining skips of the skips 0 through 27 ≡ 28 of K_{57}; for n = 8, the outer skips in K_{73} are 0, 1, 3, 7, 15, 4, 17, 8, 9; for n = 9, in K_{91} they are 0, 1, 5, 17, 21, 6, 4, 15, 3, 9; for n = 11, in K_{133} they are 6, 0, 1, 13, 11, 31, 18, 5, 4, 3, 17, 12.

3. ROTATIONAL MODELS FOR AFFINE PLANES

The Steiner system on K_9, which is AG(2,3), may be presented as the eight skip 0, 1, 2 triangles of K_8, together with the four triangles formed by joining an exterior vertex α (say α is the origin of AG(2,3)) with the four skip-3 (diagonal) edges of K_8. If the vertices of K_8 are numbered cyclically as 1 through 8, the eight skip 0, 1, 2 triangles are 124, 235, 346, 457, 568, 671, 782, 813. It is easy to verify that the Steiner system is AG(2,3): e.g., lines 124, 568, α37 are a typical bundle of three parallel lines; the triangles which include α as a vertex are the pencil of four lines through α; and through each point of K_8 there is a pencil of four lines, e.g., for vertex 1, the lines 124, 671, 813, α15.

Following is a different model for AG(2,3). Present the Steiner system on K_9 as the join of an exterior triangle (line of AG(2,3)) and K_6. The equiskip-1 triangles of K_6 are the two parallel lines to the exterior line. The three skip-2 diagonals of K_6, together with two disjoint sets of three of the skip-0 edges, are three sets of three disjoint edges; the remaining nine lines are the joins of each of the vertices of the exterior triangle to the edges of one of the disjoint sets.

The affine planes AG(2,4) and AG(2,5) have models with exterior point α, like the first given above for AG(2,3). For AG(2,4), five of the twenty lines are the joins of exterior vertex α to the five equiskip-4 triangles of K_{15}. The remaining fifteen lines are the fifteen quadruples which have exterior skips 3, 1, 0, 7 $\equiv$ 6, inner skips 2, 5. For AG(2,5), α joined to the six equiskip-5 quadruples of K_{24} forms a pencil of six lines through α. The remaining twenty-four lines are the twenty-four quintuples of K_{24} with exterior skips 0, 2, 4, 1, 10, inner skips 3, 8, 9, 7, 6. (Skips 5, 11 are covered by the equiskip-5 quadruples.)

The affine plane AG(2,5), like AG(2,3), also has a model formed by joining an exterior line of 5 points to K_{20}. The four parallel lines to the exterior line are the four equiskip-3 quintuples of K_{20}; they cover edges 7, 11. Five more lines are the five equiskip-4 quadruples with their points of intersection with the exterior line. The remaining twenty lines are the twenty quadruples of K_{20} which have outer skips (10 $\equiv$ 8), 1, 0, 5, inner skips 2, 6, joined to their points of intersection with the exterior line. Here is a detailed verification by enumeration: Denote the points of the exterior line as a, b, c, d, e, and the vertices of K_{20} (say in clockwise order) as 1,...,20. The five bundles of five parallel lines, besides the bundle first described, are the five columns of five lines, listed below.

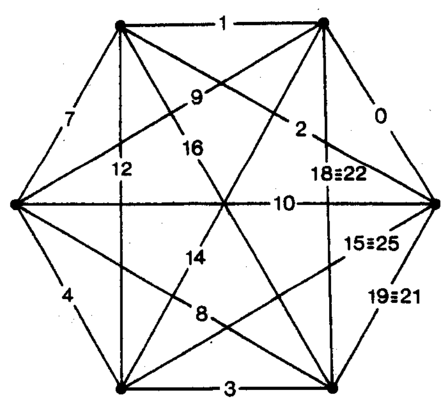
16(11)(16)a	134(10)b	245(11)c
27(12)(17)b	578(14)a	689(15)b
38(13)(18)c	9(11)(12)(18)e	(10)(12)(13)(19)a
49(14)(19)d	(13)(15)(16)2d	(14)(16)(17)3e
5(10)(15)(20)e	(17)(19)(20)6c	(18)(20)17d

356(12)d	467(13)e
79(10)(16)c	8(10)(11)(17)d
(11)(13)(14)(20)b	(12)(14)(15)1c
(15)(17)(18)4a	(16)(18)(19)5b
(19)128e	(20)239a

4. THE AFFINE PLANE AG(2,7)

Regard K_{49} as the join of K_7 and K_{42}. Then K_7 and the six equiskip-5 heptagons of K_{42} are one bundle of seven parallel lines. The seven equiskip-6 hexagons determine one bundle of seven parallel lines which intersect the exterior line, K_7, in its seven points. Together the equiskip heptagons and hexagons cover the edges of skips 5, 6, 11, 13, 17, 20 of K_{42}. The remaining forty-two lines, which also intersect K_7, are determined by the forty-two hexagons which have the remaining skips 0, 1, 2, 3, 4, 7, 8, 9, 10, 12, 14, 15, 16, 18, 19 of K_{42}, as marked in Fig. 2.

FIGURE 2



5. AFFINE PLANES OF NONPRIME-POWER ORDER

Until he discovered a slight arithmetical error, the author thought he had a model for AG(2,10), analogous to the one above for AG(2,7). That is,

K_{100} is regarded as the join of K_{10} and K_{90}. The nine equiskip-8 decagons of K_{90} and K_{10} are one bundle of ten parallel lines. The ten equiskip-9 nonagons of K_{90} determine another bundle of parallel lines which intersect the outer line, K_{10}, in its ten points. The equiskip decagons and nonagons cover the edges of skips 8, 9, 17, 19, 26, 29, 35, 39, 44. The remaining thirty-six skips of K_{90} should be covered by ninety rotational nonagons. On discovering his error, the author in haste assumed that if he could find twelve series of rotational triangles to cover the skips, the triangles could be assembled into nonagons as needed, since there is a Steiner system on K_9. He did indeed find a profusion of solutions for twelve triples (closed triangles) of skips covering the thirty-six skips, but so far has not succeeded in assembling one of the solutions for the triples into a nonagon. Computer help to decide this, and also to decide whether any of many other models for AG(2,10), AG(2,12), AG(2,15) proposed by the author are successful, has been promised by some of his colleagues who are good computer programmers.

By the Bruck-Ryser theorem, there does not exist a finite plane of order six. The edges of K_{43} are covered uniquely by the seven rotational series of triangles with skips 0, 16, 17; 1, 10, 12; 2, 15, 18; 3, 4, 8; 5, 13, 19; 6, 7, 14; 9, 11, 20. These cannot be assembled into a heptagon, for if they could be, we would have a model as in section 2 for PG(2,6). On the other hand, the nonagons in the model of PG(2,8) are dissectible into twelve rotational series of triangles which uniquely cover the edges of K_{73}. The triangles of course may be reassembled to form the nonagons.

Marshall Hall has proved that there does not exist a cyclic or rotational projective plane of order ten. The existence of a rotational affine plane with one exterior point in the lower order cases, in which cyclic projective planes exist, suggests to the author that Hall's result may imply also that there does not exist a rotational model with an exterior point for an affine plane of order ten. (Also, see [4].) But perhaps there is, as indicated above, a rotational affine model with an exterior line, exterior pair of lines, exterior pencil, or et cetera.

6. DUAL PLANES

The dual AG(2,n)* of AG(2,n) is PG(2,n) with deletion of one pencil and its "focus," but not of the other points of the lines of the pencil. In AG(2,n)*, not every two points are on a line, but every two lines are on a point. The number of lines per pencil is n (the same as the number of lines per bundle in AG(2,n)); the number of points on each line is (n + 1) (the same as the number of lines per pencil in AG(2,n)). For example, the obvious model for AG(2,3)* is the join $K_4 + K_8$: the nine lines are K_4, and the eight skip 0, 1, 2 triangles of K_8 with their points of intersection in K_4. Explicitly, if 1, 2, 3, 4 are the vertices of K_4, and 5, ..., 9, 0, α, β the vertices of K_8, then the nine lines of AG(2,3)* are 1234, 5681, 6792, 7803, 89α4, 90β1, 0α52, αβ63, β574. The twelve uncovered edges (pairs of points which are not on lines) are the diagonals 59, 60, 7α, 8β of K_8, and the eight bipartite edges 17, 28, 39, 40, 1α, 2β, 35, 46. The model for AG(2,4)* is the join $K_5 + K_{15}$. The sixteen lines are K_5 and the fifteen quadruples in K_{15} with outer skips 1, 2, 3, 5, inner skips 4, 6, together with their points of intersection in K_5. The pairs of points which are not on lines include the ends of each of the skip-0 edges.

For AG(2,5)* = $K_6 + K_{24}$, the twenty-five lines are K_6 and the twenty-four pentagons having outer skips 0, 1, 5, 10, 3, inner skips 2, 7, 6, 8, 4, with their points of intersection in K_6. The uncovered edges of K_{30} include the twenty-four skip-9 and twelve skip-11 edges in K_{24}

For the above model of AG(2,5)*, the pentagons with outer skips 1, 5, 3, 8, 2, inner skips 4, 7, 9, 10, 11, cannot serve as lines, because there are only twelve skip-11 or diagonal edges in K_{24}, so that the pairs of end-points of the diagonals each would be on two lines. These latter pentagons, however, form some kind of partial combinatorial design on twenty-four objects: the pairs corresponding to the skip-0 and skip-6 edges are not covered; the pairs corresponding to skips-1, ..., 5, 7, ..., 10 are uniquely covered; and the pairs corresponding to the diagonals are each covered twice; by the twenty-four pentagons or blocks of five objects.

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NOTES ADDED IN PROOF

Following is an easy matricial construction for AG(2,p), p prime, which shows the existence of a (graphical) model which is cyclic in a weaker sense than that of [4] and which does not require that there be an exterior point, or fixed point of a cyclic collineation. See Remark at p. 300 of [4]. In contrast with those in [4], the difference sets involved here are of a partial or mixed nature. (The author poses as a problem to find a similar matricial construction for the Galois AG(2,p^k) $k > 1$.)

For the graphical-cyclical model for AG(2,3), numerate the points of K_9 as in Fig. 3. The vertical bundle of three parallel lines consists of the three equiskip-2 triangles, i.e., the rows of

$$\begin{pmatrix} 0 & 1 & 2 \\ 10 & 11 & 12 \\ 20 & 21 & 22 \end{pmatrix}$$

The horizontal bundle I consists of the lines

$$\begin{pmatrix} 0 & 10 & 20 \\ 1 & 11 & 21 \\ 2 & 12 & 22 \end{pmatrix}$$

The next bundle II,

$$\begin{pmatrix} 0 & 11 & 22 \\ 1 & 12 & 20 \\ 2 & 10 & 21 \end{pmatrix}$$

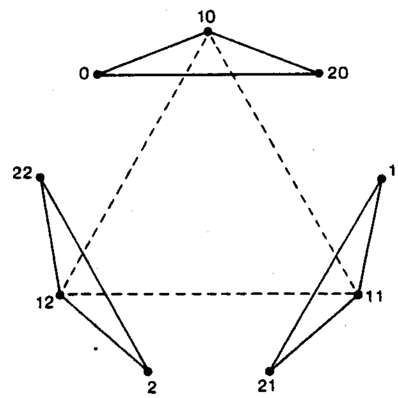
is obtained from I by taking the diagonal of I as first line, and counting cyclically to obtain two further lines.

The last bundle III,

$$\begin{pmatrix} 0 & 12 & 21 \\ 1 & 10 & 22 \\ 2 & 11 & 20 \end{pmatrix}$$

is obtained similarly from II. Referring to K_9, the lines of bundle I are three skip 0, 0, 1 triangles; the lines of bundle II are three skip 0, 3, 3 triangles; finally, the lines of III are three skip 1, 1, 3 triangles. The twelve triangles, in cycles of three, cover all of the skips 0, 1, 2, 3 ≡ 4 of K_9.

FIGURE 3



To further convey the idea, which produces a quasicyclic model for each prime p, consider AG(2,5). Disposing of excess notation and punctuation, the horizontal bundle I is

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 & 3 \\ 4 & 4 & 4 & 4 & 4 \end{pmatrix}$$

with II, III, IV, and V, respectively,

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 & 0 \\ 2 & 3 & 4 & 0 & 1 \\ 3 & 4 & 0 & 1 & 2 \\ 4 & 0 & 1 & 2 & 3 \end{pmatrix} \qquad \begin{pmatrix} 0 & 2 & 4 & 1 & 3 \\ 1 & 3 & 0 & 2 & 4 \\ 2 & 4 & 1 & 3 & 0 \\ 3 & 0 & 2 & 4 & 1 \\ 4 & 1 & 3 & 0 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 3 & 1 & 4 & 2 \\ 1 & 4 & 2 & 0 & 3 \\ 2 & 0 & 3 & 1 & 4 \\ 3 & 1 & 4 & 2 & 0 \\ 4 & 2 & 0 & 3 & 1 \end{pmatrix} \qquad \begin{pmatrix} 0 & 4 & 3 & 2 & 1 \\ 1 & 0 & 4 & 3 & 2 \\ 2 & 1 & 0 & 4 & 3 \\ 3 & 2 & 1 & 0 & 4 \\ 4 & 3 & 2 & 1 & 0 \end{pmatrix}$$

In K_{25}, the vertical bundle consists of the five equiskip-4 pentagons, which cover all edges of skips 4, 9. Bundle I consists of five 0, 0, 0, 0, 3 pentagons, which cover edges of skips 0, 1, 2, 3. Bundle II consists of five 0, 5, 5, 5, 5, pentagons, which cover edges of skips 0, 5, 6, 11. Bundle III consists of five 2, 2, 7, 2, 7 pentagons, which cover edges of skips 2, 5, 7, 10. Bundle IV consists of five 6, 1, 6, 6, 1 pentagons, which cover edges of skips 1, 6, 8, 10. Bundle V consists of five 8, 3, 3, 3, 3, pentagons, which cover edges of skips 3, 7, 8, 11. Altogether, the thirty pentagons, in six cycles of five, cover all of the skips 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11 (twenty-five edges of each skip) of K_{25}.

For a start on an AG(2,10), the five equiskip-4 decogons (= complete subgraphs K_{10}) of K_{50} may be taken as five vertical lines. These use the edge of skips 4, 9, 14, 19, 24. A system of one hundred pentagons (= complete subgraphs K_5), formed from the twenty remaining skips of 0 through 23, perhaps in twenty cycles of five pentagons each, would be equivalent to three mutually orthogonal Latin squares of side 10. Whether there are three such squares is an open research question. Mostly by experimental search (unaided by computer), the author has found systems of one hundred edge-disjoint K_5's, but unfortunately for none found so far is the remaining subgraph of K_{50} the union (edgewise) of five K_{10}'s. Neither has the author been able to muster sufficient number-theoretic resources for a proof of nonexistence (say in cycles of five), nor has the earlier-promised computer programming help from his colleagues been forthcoming.

A system of one hundred pentagons, in cycles of ten, covering the edges of the twenty skips, is equivalent to four mutually orthogonal Latin squares. The author has produced systems of quadruples of K_{40}, in cycles of five, but which give rise only to two orthogonal squares; cycles of ten would imply ten bundles of parallel 4-point segments and therefore three mutually orthogonal squares.