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ON THE CURTISS NONSINGULARITY CONDITION IN HARMONIC POLYNOMIAL INTERPOLATION*

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1. Introduction. The set L_{2n+1} of all harmonic polynomials in two real variables x, y, of degrees not exceeding n, is a real linear space of $(2n + 1)$ dimensions. With domain decreased from the x, y-plane to any finite set $Z_{2n+1} = \{z_1, \cdots, z_{2n+1}\}$ of $(2n + 1)$ points, the linear space is still of dimension $(2n + 1)$ provided that the points do not lie on any locus $u(x, y) = u(z) = 0$, $u \neq 0$, $u \in L_{2n+1}$. Call such a set Z_{2n+1} (for which the linear space is not of lower dimension) a *rank-set*. By the maximum principle, a harmonic polynomial which vanishes on an arbitrary simple closed curve C is identically zero. It can be shown consequently that for each n, there exist rank-sets of $(2n + 1)$ points on C. (See [1], [3], [4].)

If $1, u_1, v_1, \cdots, u_n, v_n$ are a basis for L_{2n+1}, the $(2n + 1) \times (2n + 1)$ matrix $A = A(Z_{2n+1})$, which has for rows $(1, u_1(z_i), v_1(z_i), \cdots, u_n(z_i), v_n(z_i))$, $z_i \in Z_{2n+1}$, $i = 1, \cdots, (2n + 1)$, is nonsingular (i.e., A has maximal rank) if and only if Z_{2n+1} is a rank-set. The reader may note a difference between the present author's formulation of matrix A and that in [1], which is analogous to the difference between the real trigonometrical and the complex exponential forms of Fourier series. The pairs of columns of a matrix A as in [1], whose entries are conjugate complex numbers, may be replaced by columns of respectively the real and imaginary parts of the complex entries, without affecting the value of the determinant of A.

The necessary and sufficient condition of Curtiss [1, p. 714] may be expressed as the requirement that the successive sets Z_{2n+1} on C, $n = 1, 2, \cdots$, be rank-sets. If and only if Z_{2n+1} is a rank-set, arbitrary real values at the points of Z_{2n+1} may be fitted by harmonic polynomials from L_{2n+1}. Let $z = \phi(w)$ be a univalent analytic map of the complement of the disc $|w| < 1$ in the w-plane onto C and its exterior, which sends the point at infinity in the w-plane into the point at infinity in the z-plane, and the circle $|w| = 1$ onto C. (More precisely, $\phi(w)$ is the univalent continuous extension of the conformal map of the exterior of the closed disc $|w| \leqq 1$. Such a mapping and extension exist by the Riemann mapping theorem and by the Osgood-Carathéodory theorem; according to the Riemann theorem, $\phi(w)$ is uniquely determined by assigning the image on C of one point of $|w| = 1$.) The images on C of the $(2n + 1)$th roots of unity on $|w| = 1$

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are known as the *Fejér points*. Curtiss asked the question whether the set F_{2n+1} of Fejér points on C is always a rank-set. The purpose of this paper is to present results on possible placements of the points such that Z_{2n+1} will and will not be a rank-set, including a negative answer to the question.

2. Some singular sets. Let a set Z_{2n+1} which is not a rank-set be called a *singular* set. A line $ax + by + c = 0$ is a harmonic polynomial curve of first degree; therefore in case all points of Z_{2n+1} are collinear, the set Z_{2n+1} is singular. Similarly if the points all lie on a hyperbola (harmonic curve of second degree), then Z_{2n+1} is singular. Call a set of n equiangularly spaced lines through a point an *n-pencil*. Obviously the zero-set of the homogeneous harmonic polynomial $\text{Im}(z^n \exp i\theta)$ is an n-pencil through the origin, which includes the line at angle θ with the x-axis. Thus Z_{2n+1} is singular if its points all lie on any n-pencil.

THEOREM 2.1. *If all* $(2n + 1)$ *points of* Z_{2n+1} *lie on a line, then* A *is of rank* $(n + 1)$; *if only* $2n$, *then* A *is of rank* $(n + 2)$. *If any* $(n + 2)$ *points of* Z_{2n+1} *lie on a line, then* Z_{2n+1} *is a singular set.*

Proof. Suppose that $z_n, z_{n+1}, \cdots, z_{2n+1}$ lie on $ax + by + c = 0$. The constant d in the conjugate harmonic polynomial $u_1 = ay - bx + d$ to $v_1 = ax + by + c$ may be chosen so that z_1, say, lies on $u_1 = 0$. Define the remaining harmonic polynomials $u_2, v_2, \cdots, u_n, v_n$ for a basis for L_{2n+1}, as homogeneous harmonic polynomials in u_1, v_1, of degrees indicated by the subscripts, such that $v_2 = 0, \cdots, v_n = 0$ on $v_1 = 0$, and $u_3 = 0, u_5 = 0, \cdots, u_n$ or $u_{n-1} = 0$ on $u_1 = 0$. That this is possible follows from the above discussion of n-pencils. On the line $v_1 = 0$, harmonic polynomials of degree n or less reduce to polynomials in u_1 of degree not exceeding n. Arbitrary values at the $(n + 2)$ points on $v_1 = 0$ cannot be interpolated by polynomials from the $(n + 1)$-dimensional linear space which is spanned by $1, u_1, \cdots, u_1^n$ (see [3]); therefore Z_{2n+1} must be singular.

In case all $(2n + 1)$ points lie on $v_1 = 0$, the dimension of the subspace of L_{2n+1} of the polynomials which vanish on $v_1 = 0$ is n; the rank of A is the complementary dimension $(n + 1)$. If $2n$ points are on $v_1 = 0$, but $v_1(z_1) \neq 0$, then the values of a harmonic polynomial from L_{2n+1} at $z_{n+1}, \cdots, z_{2n+1}$ uniquely determine the values at $z_2, \cdots, z_r$. Thus the maximum possible number of points in a rank-set for L_{2n+1} restricted to Z_{2n+1} is $(n + 2)$. The points $z_1, z_{n+1}, \cdots, z_{2n+1}$ form such a rank-set. Therefore the rank of A is $(n + 2)$.

THEOREM 2.2. *If all* $(2n + 1)$ *points lie on a pair of perpendicular lines, then the rank of* A, *for even* n, *does not exceed* $(3n/2 + 1)$, *and for odd* n, *does not exceed* $(\frac{3}{2})$ $(n + 1)$.

Proof. To establish this theorem, we note that $v_2, v_4, \cdots, v_{n-1}$ or v_n vanish both on $v_1 = 0$ and on $u_1 = 0$. Thus the subspace of L_{2n+1} of all polynomials which vanish on the pair of perpendicular lines has dimension at least $(n - 1)/2$ or $n/2$. The complementary dimension, which is the rank of A, therefore is at most $(2n + 1) - (n - 1)/2$ or $(2n + 1) - n/2$.

THEOREM 2.3. *If all* $(2n + 1)$ *points lie on a pair of conjugate harmonic polynomial curves of degree* k, $k \leqq n/2$ *or* $(n - 1)/2$, *and* ν *is the largest integer such that* $2\nu k \leqq n$, *then the rank of* A *does not exceed* $(2n + 1) - \nu$.

Proof. By hypothesis, Z_{2n+1} lies entirely on the harmonic polynomial locus of $uv = 0$, which is of degree $2k$, since the conjugate harmonic polynomials u, v are of degree k. If $U_m = \operatorname{Re}(u + iv)^m$, $V_m = \operatorname{Im}(u + iv)^m$, then for each $m = 1, \cdots, \nu$, the harmonic polynomial $U_m V_m$, which is of degree $2m$, contains uv as a factor. Therefore Z_{2n+1} lies on $U_m V_m = 0$. The $U_m V_m$ are ν linearly independent harmonic polynomials which vanish on Z_{2n+1}; therefore the complementary dimension does not exceed $(2n + 1) - \nu$.

If a set Z_{2n+1} is symmetric with respect to the real axis, then it must contain an odd number of points on the real axis, and in particular it must contain at least one real point. A harmonic polynomial curve is real-symmetric if it is of the form $\operatorname{Re} P(z) = 0$, where $P(z)$ is a polynomial in z having real coefficients, but a real-symmetric harmonic polynomial curve need not be of this form. The n-pencil $\operatorname{Re}(iz^n) = 0$, for example, is real-symmetric. Denote by $\bar{Q}$ the polynomial whose coefficients are the complex conjugates of those of Q. If Z_{2n+1} is singular, then the points of Z_{2n+1} lie on $\operatorname{Re} Q(z) = 0$, where $Q(z)$ is a polynomial of degree not exceeding n. If also Z_{2n+1} is real-symmetric, then $\operatorname{Re} \bar{Q}(\bar{z}_j) = \operatorname{Re} Q(z_j) = 0$ for $z_j \in Z_{2n+1}$; thus the points of Z_{2n+1} are on $\operatorname{Re} \bar{Q}(z) = 0$ as well as on $\operatorname{Re} Q(z) = 0$. Therefore Z_{2n+1} lies on $\operatorname{Re}[Q(z) + \bar{Q}(z)] = 0$. Unless $Q(z) + \bar{Q}(z)$ is identically zero, which occurs if and only if $Q(z)$ has only pure imaginary coefficients, the singular real-symmetric set Z_{2n+1} lies on $\operatorname{Re} P(z) = 0$, where $P = Q + \bar{Q}$ has real coefficients. For $v_h = \operatorname{Im} z^h$, $v_h(\bar{z}_j) = -v_h(z_j)$; thus in either case a singular real-symmetric Z_{2n+1} lies on a real symmetric harmonic polynomial curve of degree not exceeding n. If a real-symmetric set Z_{2n} of $2n$ points lies on $\operatorname{Re} Q(z) = 0$, where $Q(z)$ has pure imaginary coefficients, then since $v_1, \cdots, v_n$ vanish on the x-axis, we have that all sets $Z_{2n+1} = Z_{2n} \cup \{x_{2n+1}\}$, for arbitrary real x_{2n+1}, are singular.

THEOREM 2.4. *Any real-symmetric set* Z_{2n} *of* $2n$ *points, without real points, lies on a curve* $\operatorname{Re} P(z) = 0$, *where* $P(z)$ *is a nontrivial polynomial with real coefficients, of degree not exceeding* n. *If* Z_{2n} *does not also lie on a curve* $\operatorname{Re} Q(z) = 0$, *then* $Z_{2n+1} = Z_{2n} \cup \{x_{2n+1}\}$ *is a rank-set for all real* x_{2n+1} *except real zeros of* P *such that* Z_{2n} *lies on* $\operatorname{Re} P(z) = 0$. *If all such* P *have no real zeros, then* Z_{2n+1} *is a rank-set for every* x_{2n+1}.

Proof. Define $u_h = \operatorname{Re} z^h$. Then the harmonic polynomials $1, u_1, \cdots, u_n$ span an $(n + 1)$-dimensional subspace of L_{2n+1}. Consequently, by Theorem 1 of [3], there exists a nontrivial real linear combination of $1, u_1, \cdots, u_n$ which vanishes at the n points $z_1, \cdots, z_n$. The coefficients of the linear combination are the coefficients of the required polynomial $P(z)$. For since by construction $z_1, \cdots, z_n$ lie on $\operatorname{Re} P(z) = 0$, we have $\operatorname{Re} P(\bar{z}_j) = \operatorname{Re} \bar{P}(z_j) = \operatorname{Re} P(z_j) = 0, j = 1, \cdots, n$. If Z_{2n+1} is singular, it lies on $\operatorname{Re} R(z) = 0$, where R is a polynomial of degree not greater than n; by hypothesis R does not have only pure imaginary coefficients. Therefore $P = R + \bar{R}$, which has real coefficients, is nontrivial, and we have $\operatorname{Re} P(x_{2n+1}) = 0$, $P(x_{2n+1}) = 0$ since $v_1, \cdots, v_n$ vanish on the real axis. Unless there is a P which vanishes at x_{2n+1}, Z_{2n+1} cannot be singular.

THEOREM 2.5. *If a real-symmetric set Z_{2n+1} contains more than one real point, it is singular.*

Proof. Suppose for example that Z_{2n+1} contains three real points x_{2n-1}, x_{2n}, x_{2n+1}. The determinant of $A(Z_{2n+1})$ is invariant under translation of the z-plane; translate so that $x_{2n+1} = 0$. Let the real-symmetric pairs of points of Z_{2n+1} be $z_1, \bar{z}_1, \cdots, z_{n-1}, \bar{z}_{n-1}$. Then since $u_j(\bar{z}) = u_j(z)$, $v_j(\bar{z}) = -v_j(z)$, A has rows $(1, u_1(z_1), v_1(z_1), \cdots, u_n(z_1), v_n(z_1))$; $(1, u_1(z_1), -v_1(z_1), \cdots, u_n(z_1), -v_n(z_1))$; $\cdots$; $(1, u_1(x_{2n-1}), 0, \cdots, u_n(x_{2n-1}), 0)$; $(1, u_1(x_{2n}), 0, \cdots, u_n(x_{2n}), 0)$. By adding each of the first $(n - 1)$ odd-numbered rows to the following even-numbered row, we see that A is singular or not according to whether the $2n \times 2n$ matrix

$$\begin{bmatrix} u_1(z_1) & v_1(z_1) & u_2(z_1) & v_2(z_1) & \cdots & u_n(z_1) & v_n(z_1) \\ u_1(z_1) & 0 & u_2(z_1) & 0 & \cdots & u_n(z_1) & 0 \\ u_1(z_2) & v_1(z_2) & u_2(z_2) & v_2(z_2) & & u_n(z_2) & v_n(z_2) \\ \vdots & & & & & & \\ u_1(z_n) & 0 & u_2(z_n) & 0 & & u_n(z_n) & 0 \\ u_1(x_{2n-1}) & 0 & u_2(x_{2n-1}) & 0 & & u_n(x_{2n-1}) & 0 \\ u_1(x_{2n}) & 0 & u_2(x_{2n}) & 0 & & u_n(x_{2n}) & 0 \end{bmatrix}$$

is singular. By expanding the determinant of the displayed matrix according to the last column, then the $(2n - 1) \times (2n - 1)$ determinants in terms of their next-to-last columns, and so on, the determinant is seen to be a linear combination of 5×5 determinants, typically

$$\begin{vmatrix} u_1(z_{n-2}) & 0 & u_2(z_{n-2}) & 0 & u_3(z_{n-2}) \\ u_1(z_{n-1}) & v_1(z_{n-1}) & u_2(z_{n-1}) & v_2(z_{n-1}) & u_3(z_{n-1}) \\ u_1(z_{n-1}) & 0 & u_2(z_{n-1}) & 0 & u_3(z_{n-1}) \\ u_1(x_{2n-1}) & 0 & u_2(x_{2n-1}) & 0 & u_3(x_{2n-1}) \\ u_1(x_{2n}) & 0 & u_2(x_{2n}) & 0 & u_3(x_{2n}) \end{vmatrix}.$$

Each of the 5×5 determinants of this form is zero; therefore A is singular. A similar argument establishes singularity of any real-symmetric Z_{2n+1} which has more than three real points.

Singularity of a set Z_{2n+1} of course is a rotational (as well as translational) invariant. Therefore Theorem 2.5 implies that any set Z_{2n+1}, which is symmetric with respect to a line containing three or more points of the set, is singular.

3. Triangularization and reduction of matrix A. A set Z_{2n+1} is a rank-set if and only if the matrix A is triangularizable in the following way, with non-zero diagonal entries. For one of $u_1, v_1, \cdots, u_n, v_n$, say r_1, we have $r_1(z_2) \neq r_1(z_1)$, or else A is singular. Let the second column of A be $r_1(z_1), \cdots, r_1(z_{2n+1})$, and subtract $r_1(z_1)$ times the first column (all 1's) from the second column, to obtain a new second column $p_1(z_1), \cdots, p_1(z_{2n+1})$, with $p_1(z_1) = 0$, $p_1(z_2) \neq 0$. Unless A is singular, another of $u_1, \cdots, v_n$, say r_2, is such that the 3×3 determinant with rows $1, 0, r_2(z_1)$; $1, p_1(z_2), r_2(z_2)$; $1, p_1(z_3), r_2(z_3)$; is not zero. Let $r_2(z_1), \cdots$ be the third column of A. Subtract a linear combination of the first column and of $p_1(z_1), \cdots$ from the third column, to obtain a new third column $p_2(z_1), \cdots$, with $p_2(z_1) = 0$, $p_2(z_2) = 0$, $p_2(z_3) \neq 0$. Continuing, the matrix A is either reduced to triangular form, with $p_j(z_{j+1}) \neq 0, j = 1, \cdots, 2n$, or it is singular.

If Z_{2n+1} is a rank-set for L_{2n+1}, then for each subspace L_{2k+1} of L_{2n+1}, $1 \leqq k < n$, there is a subset Z_{2k+1} of $(2k+1)$ points of Z_{2n+1} which is a rank-set for L_{2k+1} (see [4]). Consequently by suitable renumeration of the points of Z_{2n+1}, it is possible to arrange so that in the triangular matrix A, p_1 and p_2 are linear combinations of $1, u_1, v_1$; p_3 and p_4 of $1, u_1, v_1, u_2, v_2$; $\cdots$; p_{2n-1} and p_{2n} of $1, u_1, v_1, \cdots, u_n, v_n$.

In the case of a real-symmetric $Z_{2n+1} = \{z_1, \bar{z}_1, \cdots, z_n, \bar{z}_n, x_{2n+1}\}$, it may be seen by use of elementary transformations and of Laplace's expansion of a determinant, that the determinant of the $(2n+1) \times (2n+1)$ matrix A reduces to that of the following product of $n \times n$ matrices:

$$\begin{bmatrix} u_1(z_1) - u_1(x_{2n+1}) & \cdots & u_n(z_1) - u_n(x_{2n+1}) \\ \vdots & & \vdots \\ u_1(z_n) - u_1(x_{2n+1}) & \cdots & u_n(z_n) - u_n(x_{2n+1}) \end{bmatrix} \cdot \begin{bmatrix} v_1(z_1) & \cdots & v_n(z_1) \\ \vdots & & \vdots \\ v_1(z_n) & \cdots & v_n(z_n) \end{bmatrix},$$

where $u_k = \operatorname{Re} z^k$, $v_k = \operatorname{Im} z^k$, $k = 1, \cdots, n$. A relation of linear dependence between the columns of the second matrix implies that $z_1, \cdots, z_n$ satisfy $\operatorname{Re} Q(z) = 0$, where Q is of degree not exceeding n and has pure imaginary coefficients; and conversely. Singularity of the first matrix is equivalent to $z_1, \cdots, z_n, x_{2n+1}$ satisfying $\operatorname{Re} P(z) = 0$, where P is of degree not exceeding n and has real coefficients. It is thus evident again that the matrix A is singular if and only if either $z_1, \cdots, z_n$ satisfy $\operatorname{Re} Q(z) = 0$, or $z_1, \cdots, z_n, x_{2n+1}$ satisfy $\operatorname{Re} P(z) = 0$, or both. Whether A is singular or not, as already noted in Theorem 2.4, there always exists a $P(z)$ of degree n or less, such that $z_1, \bar{z}_1, \cdots, z_n, \bar{z}_n$ lie on the locus $\operatorname{Re} P(z) = 0$.

4. Finite star-sets. Sets of $2n + 1$ points of the three following general types seemed to the author to be likely candidates for rank-sets. For a set S_{2n+1}, an "equiangular star-set", point z_k is arbitrarily located on a positive $(r > 0)$ ray $r \exp ik2\pi/(2n + 1)$, for each $k = 1, \cdots, 2n + 1$. For a set T_{2n+1}, $2n$ points are located in pairs on n lines of an $(n + 1)$-pencil, one pair on each line, with the origin in between the points of the pair; the $(2n + 1)$th point $z_{2n+1} \neq 0$ is on the $(n + 1)$th line. Finally for a set R_{2n+1}, $z_1, \cdots, z_{2n}$ are located in pairs, with the origin between, on the n lines of an n-pencil; z_{2n+1} is any point off the n-pencil. Denote by R_{2n} the subset of the $2n$ points which are on the n-pencil. Suppose that R_{2n} lies on a harmonic polynomial locus $q = 0$, as well as on the n-pencil $v_n = 0$, where q is not a scalar multiple of v_n. Then for z_{2n+1} not a simultaneous zero of q and v_n, R_{2n+1} lies on the locus $q(z_{2n+1})v_n(z) - v_n(z_{2n+1})q(z) = 0$. Thus a set R_{2n+1} is a rank-set if and only if the subset R_{2n} of points on the n-pencil does not lie on any other harmonic polynomial locus of degree n or lower.

Clearly for $n = 1$ all sets of each of the above three types are nonsingular. For $n = 2$, however, it can be shown that there are sets T_5 which lie on a proper equilateral hyperbola (as well as on the 3-pencil $y(y^2 - 3x) = 0$); for such a T_5, the matrix $A(T_5)$ is singular. Likewise it can be shown that there are equiangular star-sets S_5 which lie on an equilateral hyperbola; for such an S_5, the matrix $A(S_5)$ is singular.

THEOREM 4.1. *Any set R_5 is a rank-set. There are sets R_6, however, which lie on another harmonic polynomial locus of degree* 3 *besides* $y(y^2 - 3x) = 0$; *for any corresponding R_7, the matrix $A(R_7)$ is singular.*

Proof. To show that the matrix $A(R_5)$ is always nonsingular, we shall construct a basis $1, g_1, h_1, g_2, h_2$ for L_5 such that $A(R_5)$ is triangular with nonzero diagonal entries. By the description of R_5, the points $z_1, \cdots, z_4$ lie on the 2-pencil $v_2 = xy = 0$. Let $h_1 = ax + by + c = 0$ pass through z_1, z_2, and the conjugate $g_1 = bx - ay + d$ pass through z_1. The point z_3 cannot be collinear with z_1, z_2; therefore $h_1(z_3) \neq 0$. Because of the location of $z_1, \cdots, z_4$ on $v_2 = 0$, the perpendicular line to $h_1 = 0$ through z_3 cannot contain z_4. Thus if $g_2 = 0$ is the equation of the 2-pencil (pair of perpendicular lines just mentioned) through z_1, z_2, z_3, we have that $g_2(z_4) \neq 0$. Finally, we have $h_2(z_5) = v_2(z_5) \neq 0$, since z_5 is off the pencil $v_2 = 0$. The matrix $A(R_5)$ is triangular with nonzero diagonal entries. Therefore, as required, the matrix is nonsingular.

We show that there are sets R_6 which are on another third degree harmonic polynomial locus besides $y(y^2 - 3x) = 0$, and which are symmetric with respect to the real axis. Consider the locus $\operatorname{Re}[(z + 1)(z - r)(z - d)] = (x + 1)[(x - r)(x - d) - y^2] - (2x - r - d)y^2 = 0$, where r and d are real. Let $z = -1$ and $z = r > 0$ be two points of R_6; then it will be sufficient to show that with $y = 3^{1/2}x$ in the above equation, there exists $r > 0$ for which the cubic in x has both a positive and a negative solution for x. The cubic is

$$[(x - r)(x - d) - 3x^2](x + 1) - 3x^2(2x - r - d) = 0.$$

Putting $(r + d) = -2$, $x = -1$ is a solution; choose $z = -1 \pm 3^{1/2}i$ as two further points of R_6. The cubic then factors as $(x + 1)\cdot(-8x^2 + 2x + rd)$, where $r + d = -2$. The quadratic factor has positive zeros for r in the range $0 < r < (3\cdot 2^{1/2}/4) - 1 = 0.0605$. If s is either of the positive zeros, then $z = s(1 \pm 3^{1/2}i)$ are possible choices for the final two points of R_6. Also of course $s(1 + 3^{1/2}i)$ and $t(1 - 3^{1/2}i)$, where t is the other positive zero, may be chosen. (For the choice $r + d = -2$, the locus $y^2 = (x + 1)(x - r)(x - d)/(3x + 1 - r - d)$ consists of a hyperbola and the line $x = -1$.)

5. Real-symmetric univalent mappings. For some choices of coefficients $d, c_0, \cdots, c_m$, univalent conformal mappings of the exterior of $|w| = 1$ are given by the algebraic functions

$$z = \phi(w) = dw + c_0 + \frac{c_1}{w} + \cdots + \frac{c_m}{w^m}, \tag{1}$$

$d \neq 0$. If the coefficients are real, the image C of $|w| = 1$ is real-symmetric. As pointed out to the author by Curtiss (at the same time that he asked the question about the Fejér points), in order that the image of $|w| = 1$ under $\phi(w)$ be a simple closed curve C, with $\phi(w)$ a mapping as described in §1, it is necessary that the coefficients in $\phi(w)$ satisfy the Gronwall-Bieberbach area condition

$$\sum_1^m k\left|\frac{c_k}{d}\right|^2 < 1. \tag{*}$$

It is necessary further that $\phi(w)$ satisfy the SCHLICHTNESS CONDITION (**): *that the derivative $\phi'(w)$ has no zeros outside of the circle $|w| = 1$.* (See [2], especially Chapter 17.) In case there are no zeros outside of or on $|w| = 1$, then the image C of $|w| = 1$ is an analytic simple closed curve. For $\phi(w)$ given by (1), the zeros of $\phi'(w)$ are those of the polynomial

$$w^{m+1}\phi'(w) = dw^{m+1} - c_1w^{m-1} - 2c_2w^{m-2} - \cdots - mc_m. \tag{2}$$

For $n = 1$, by definition the set F_3 of Fejér points consists of the images $z_1 = \phi(\omega)$, $z_2 = \phi(\bar{\omega})$, $z_3 = \phi(1)$ of the cube roots of unity, $\omega, \bar{\omega}, 1$. Let $m = 3$. Then on $|w| = 1$, the effect of $z = \phi(w)$ is the same as that of the polynomial mapping $z = dw^4 + c_0w^3 + c_1w^2 + c_2w + c_3$. We have $\phi(1) = d + c_0 + c_1 + c_2 + c_3$, $\phi(\omega) = d\omega + c_0 + c_1\bar{\omega} + c_2\omega + c_3$,

$\phi(\bar{\omega}) = d\bar{\omega} + c_0 + c_1\omega + c_2\bar{\omega} + c_3$. With real coefficients, for $z_1 = \phi(\omega)$ to be real, we must have $d + c_2 = c_1$, but this forces z_2 to be equal to z_1, so that ϕ is not univalent. Clearly $\phi(\bar{\omega})$ is the complex conjugate of $\phi(\omega)$. Therefore to have z_1, z_2, z_3 on a line, as required for $A(F_3)$ to be singular, let us choose c_0 so that the image of 1 is $z_3 = 0$, and z_2 and z_1 are on the imaginary axis. This will be the case if the coefficients satisfy the relations $(d + c_1 + c_2) = -(c_0 + c_3)$, $(d + c_1 + c_2) = 2(c_0 + c_3)$, or $d + c_1 + c_2 = 0$, $c_0 + c_3 = 0$. Put $d = 1$, $-3c_3 = c$, $-c_1 = b$. Then the function $\phi(w)$ assumes the form

$$\phi(w) = w + \frac{c}{3} - \frac{b}{w} - \frac{(1-b)}{w^2} - \frac{c}{3w^3}. \tag{3}$$

The area condition becomes

$$b^2 + 2(1-b)^2 + \frac{c^2}{3} < 1. \tag{3*}$$

Putting the same values in the polynomial (2), we obtain

$$w^4 + bw^2 + 2(1-b)w + c = 0 \tag{3**}$$

as the polynomial equation whose roots must lie inside the closed disc $|w| \leqq 1$.

For $n = 2$, according to Theorem 2.1 or 2.2, the set of Fejér points F_5 will be singular if the four images $z_1 = \phi(\alpha)$, $z_2 = \phi(\alpha^2)$, $z_3 = \phi(\bar{\alpha})$, $z_4 = \phi(\bar{\alpha}^2)$, where $\alpha = \cos 72° + i \sin 72°$ is a complex fifth root of unity, are on the imaginary axis. With $m = 3$, using (1) and putting $c_0 = 0$, we see that $\phi(\alpha) = d\alpha + c_1\bar{\alpha} + c_2\bar{\alpha}^2 + c_3\alpha^2$, $\phi(\alpha^2) = d\alpha^2 + c_1\bar{\alpha}^2 + c_2\alpha + c_3\bar{\alpha}$; these are on the imaginary axis if

$$d = -c_1, \qquad c_2 = -c_3. \tag{4}$$

By real symmetry, $\phi(\bar{\alpha})$, $\phi(\bar{\alpha}^2)$ will be pure imaginary if $\phi(\alpha)$, $\phi(\alpha^2)$ are. In order to possibly satisfy the area condition (*), we cannot have $|c_1/d| = 1$; therefore we add $c_4/w + c_4/w^2 + c_4/w^3 + c_4/w^4$ to (1) with $m = 3$. We have $\alpha^4 + \alpha^3 + \alpha^2 + \alpha = -1$; therefore the sum of the added terms is $-c_4$, and similarly for $w = \alpha^2, \alpha^3, \alpha^4$. Putting $d = 1$, $c_0 = c_4$, $c_2 = a$, we have that the mapping

$$z = \phi(w) = w + c_4 + \frac{c_4 - 1}{w} + \frac{c_4 + a}{w^2} + \frac{c_4 - a}{w^3} + \frac{c_4}{w^4} \tag{5}$$

will place four points of F_5 on the imaginary axis; the fifth point, $\phi(1) = 5c_4$, is on the real axis. Also $\phi(-1) = c_4 + 2a$; clearly in order that C be a simple closed curve, it is necessary that $\phi(-1) < \phi(1)$, or $a < 2c_4$. In terms of c_4 and a, the area condition (*) for (5) reduces to

$$10c_4^2 + 5a^2 < 2c_4(1 + a). \tag{5*}$$

For schlichtness, the roots of the polynomial equation

$$(5^{**})\qquad w^5 + (1 - c_4)w^3 - 2(c_4 + a)w^2 - 3(c_4 - a)w - 4c_4 = 0$$

must be inside $| w | \leqq 1$.

For $n = 3$, by Theorem 2.1 the matrix $A(F_7)$ will be singular if five of the Fejér points are on the imaginary axis. Consider $\phi(w) = w - 1/w - a/w^2 + a/w^5$. Let α be the basic seventh root of unity. Then $\phi(1) = 0$, and $\phi(\alpha)$, $\phi(\alpha^2)$, $\phi(\bar{\alpha})$, $\phi(\bar{\alpha}^2)$ are on the imaginary axis, as required. Also $\phi(\alpha^3)$, $\phi(\bar{\alpha}^3)$ are on the imaginary axis. Again in order to be able to satisfy the area condition, we cannot have $| c_1/d | = 1$, so to obtain the simplest possible form for $\phi(w)$, we add $b/w - b/w^6$. This is permissible since it adds zero to $\phi(1)$, and only pure imaginaries to $\phi(\alpha)$, $\phi(\alpha^2)$. The resulting function $\phi(w)$ is

$$(6)\qquad \phi(w) = w - \frac{(1-b)}{w} - \frac{a}{w^2} + \frac{a}{w^5} - \frac{b}{w^6}.$$

The area condition reduces to

$$(6^*)\qquad 7(a^2 + b^2) < 2b.$$

For schlichtness, the roots of the polynomial equation

$$(6^{**})\qquad w^7 + (1 - b)w^5 + 2aw^4 - 5aw + 6b = 0$$

must be inside the closed unit disc.

6. Coefficient region for quadratic roots inside a circle. Elementary calculations show that the roots of the quadratic equation $\zeta^2 + b\zeta + c = 0$ are inside the unit circle provided the point (b, c) is inside the triangle in the b, c-plane which has vertices $(-2, 1)$, $(0, -1)$, $(2, 1)$. The roots are complex, of modulus $\leqq 1$, if (b, c) is in the region bounded by the parabola $c = b^2/4$ and by the line $c = 1$. One real root is 1 if (b, c) is on the edge of the triangle which is contained in the line $b + c = -1$, and one root is -1 if (b, c) is on the edge which is contained in $b - c = 1$. The roots are complex, of modulus 1, if (b, c) is on the edge contained in $c = 1$. In case of complex $c = \rho \exp i2\psi$ with real b, the modulus r of the root $r \exp i\theta$ of larger modulus is $\leqq 1$ if

$$b^2[(\sin(\psi - \theta) + \sin\theta)^2 - 2(1 - \cos\psi)\sin\theta\sin(\psi - \theta)] \leqq 2\sin(\psi - \theta).$$

For this it is necessary that $\rho \leqq 1$, $| b | < 1 + \rho$. For roots of equal modulus, we must have $| b | \leqq 2\rho^{1/2} \leqq 1 + \rho$. If $F(w)$ is a polynomial function of w, of the form $(w^2 + \beta w)^2 + b(w^2 + \beta w) + c$, with (b, c) inside the triangle, and if the inequality just stated, with β in place of b, is satisfied, then the zeros of $F(w)$ are inside the unit disc. For (b, c) inside the triangle, for any positive integer k, the zeros of $(w^{2k} + bw^k + c)$ are inside the unit disc.

For real b and c, the minima and maxima of $|\zeta^2 + b\zeta + c| = |\exp i\theta + b + c\exp(-i\theta)|$ are included among the three values $|1 + c - b|, |1 + c + b|, (|1 - c|)(|b^2 - 4c|)^{1/2}/2|c|^{1/2}$. For use in the next section, we recall to the reader Rouché's theorem, which applied to polynomial functions on the unit disc states that if $|P(w)| < |Q(w)|$ on $|w| = 1$, then $F(w) = P(w) + Q(w)$ has the same number of zeros in the open disc $|w| < 1$ as does $Q(w)$.

Following is an indication of a possible method of enlarging the class of univalent mappings which may be considered, without increasing the difficulty of deciding whether the schlichtness condition (**) is satisfied. The polynomial

$$(7) \quad F(w) = \frac{dw^{m+2}}{m+2} - \frac{c_1}{m} w^m - \frac{2c_2}{(m-1)} w^{m-1} - \cdots - mc_m + f,$$

where f is an arbitrary constant, has the polynomial (2) as its derivative. If for some choice of f, $F(w)$ assumes the form $G(P(w))$, where $G'(\zeta)$ has its zeros in the disc $|\zeta| \leqq g$, and $P(w) = w^3 + b_0w^2 + b_1w + b_2$ is known to be such that the zeros of $P(w) - \rho\exp i\theta$ are inside $|w| \leqq 1$ for $\rho \leqq g$, then $F'(w) = w^{m+1}\phi'(w) = G' \cdot P'(w)$ also has its zeros inside $|w| \leqq 1$ if the point $(2b_0/3, b_1/3)$ is inside the quadratic coefficient triangle.

7. Singular Fejér sets, $n = 1$. If a function $\phi(w)$ satisfies the area and schlichtness conditions (*) and (**), then so does $\phi(w\exp i\theta)$ for each θ. The determinant of the matrix $A(F_3; \theta)$ is a continuous function of θ. The "backward" Fejér points $\phi(-1), \phi(-\bar{\omega}), \phi(-\omega)$ correspond to $\theta = \pi$. In case the determinants $|A(F_3; 0)|$ and $|A(F_3; \pi)|$ have opposite signs, by continuity there is an intermediate value of θ for which $|A(F_3; \theta)| = 0$. Clearly $|A(F_3; 0)|$ and $|A(F_3; \pi)|$ have opposite signs if $[\operatorname{Re}\phi(\omega) - \phi(1)]$ and $[\operatorname{Re}\phi(-\omega) - \phi(-1)]$ have the same sign. Thus for any permissible $\phi(w)$, there will be a singular F_3 on C if the product $[\operatorname{Re}\phi(\omega) - \phi(1)] \cdot [\operatorname{Re}\phi(-\omega) - \phi(-1)]$ is nonnegative.

Consider the mapping

$$(8) \qquad z = \phi(w) = w - \frac{b}{w} + \frac{c_2}{w^2} - \frac{c}{3w^3},$$

which is essentially the most general of the real-symmetric form which is possible with $m = 3$. For this $\phi(w)$, the above product has the value $(\frac{9}{4})[c_2^2 - (1-b)^2]$. To satisfy the area condition

$$(8^*) \qquad b^2 + 2c_2^2 + c^2/3 < 1,$$

and the requirement that the product be nonnegative, in the b, c_2-plane the point (b, c_2) must be inside the ellipse $b^2 + 2c_2^2 = (1 - c^2/3)$, and also above the line $c_2 = -(b - 1)$, or below the line $c_2 = (b - 1)$. This is possible if b is in the interval $[2 - (1 - c^2)^{1/2}]/3 \leqq b \leqq [2 + (1 - c^2)^{1/2}]/3$, and c_2 is in the interval $[1 - (1 - c^2)^{1/2}]/3 \leqq c_2 \leqq [1 + (1 - c^2)^{1/2}]/3$.

THEOREM 7.1. *To each point* (b, c_2, c) *in a three-dimensional region in the coefficient-space of* b, c_2, c *for* $z = \phi(w)$ *as in* (8), *there corresponds an analytic simple closed curve* C, *the image of* $| w | = 1$, *on which there is a singular Fejér set* F_3.

Proof. For schlichtness, the roots of

$$w^4 + bw^2 - 2c_2w + c = 0 \tag{8**}$$

must be inside the unit disc. Ferrari's resolvent cubic for (8**) is $(s - b) \cdot (s^2 - 4c) - 4c_2^2 = 0$. (See [5, p. 50].) The quadratic factors of the quartic are $w^2 - \beta x + (s/2 - c_2/\beta)$, $w^2 + \beta x + (s/2 + c_2/\beta)$, where s is any root of the cubic, and $\beta = (s - b)^{1/2}$. For $b \leqq 2c^{1/2}$ and any $c_2 \neq 0$, the cubic has a root greater than $2c^{1/2}$. At $(b, c_2, c) = (\frac{4}{5}, (0.044)^{1/2}, \frac{1}{4})$, the root is $s = 1.2$, and $0.63 < \beta = (0.4)^{1/2} < 0.64$. The points $(\beta, s/2 \pm c_2/\beta)$ are inside the quadratic coefficient triangle. The values $b = 0.8$ and $c_2 = 0.21^-$ are in the required intervals, and also (b, c_2) is inside the area-ellipse. It is clear that the required region exists, and has $(\frac{4}{5}, 0.21^-, \frac{1}{4})$ as an interior point.

To show that there exists a real-symmetric singular F_3, we impose the requirement $c_2 = -(1 - b)$, as was discussed at the beginning of §5. Applying Rouché's theorem to (3**), with $Q(w) = w^4 + bw^2 + c$, $P(w) = 2(1 - b)w$, we see that the roots are inside the unit disc provided that

$$2(| 1 - b |) < \min\{| 1 + b + c |, | 1 - b + c |, (| 1 - c |)| (4c - b^2)/4c |^{1/2}\}.$$

This requirement, as well as the area condition (3*), is satisfied for example by $b = 0.8$, $c = 0.25$.

The mapping

$$z = \phi(w) = 7w - 1 - \frac{4}{w} - \frac{2}{w^2} + \frac{1}{w^3} - \frac{1}{w^4} \tag{9}$$

satisfies the area and schlichtness conditions (*), (**), and places F_3 on the imaginary axis. Since the zeros of the corresponding polynomial (2) are -1, $\pm i$, $(\frac{1}{2} \pm 7^{1/2} \cdot 3i/14)$, the simple closed curve C which is the image of $| w | = 1$ is not analytic, but has "corners" at the points $z = -8$, $\pm 12i$, which correspond to the three zeros of unit modulus.

8. Nonsingularity of $A(C; F_3)$ for $m = 2$. The (essentially) most general mapping $z = \phi(w)$ with $m = 2$ may be written in the form

$$z = \phi(w) = w - \frac{p}{w} - \frac{q}{2w^2}. \tag{10}$$

For this $\phi(w)$, the above product has the value $(\frac{9}{4})[q^2/4 - (1-p)^2]$. To satisfy the area condition $p^2 + q^2/2 < 1$, and the requirement that the product be nonnegative, in the p, q-plane the point (p, q) must be either in the region above the line $q = -4(p-1)$, bounded by the line and the ellipse $p^2 + q^2/2 = 1$, or in the region below the line $q = 4(p-1)$, bounded by the line and the ellipse. The lines intersect the ellipse in the points $(1, 0)$ and $(\frac{7}{9}, \pm\frac{8}{9})$. For schlichtness, the solutions of

$$w^3 + pw + q = 0 \tag{10**}$$

must be inside the unit disc. By Tartaglia's formulas (plagiarized by Cardan), (see [5, p. 46]), the solutions of (10**) are $D + E$, $D\omega + E\bar{\omega}$, $D\bar{\omega} + E\omega$, where $D^3 = (-q/2 + R^{1/2})$, $E^3 = (-q/2 - R^{1/2})$, $R = (p/3)^3 + (q/2)^2$. The values $p = \frac{8}{9}$, $q = \frac{4}{9}$ satisfy the area and product conditions, but $D + E > \frac{13}{9}$. Also the values $p = \frac{25}{27}$, $q = \frac{8}{27}$ are in one of the de- scribed regions bounded by a line and the ellipse, but $D + E > \frac{10}{9}$. Ap- parently it is not possible for the image C of $|w| = 1$, under a real-sym- metric $\phi(w)$ with $m = 2$, to contain a singular F_3. For $p = \frac{2}{3}$, $q = \frac{2}{27}$, $D^3 = \frac{2}{27}$, $E^3 = -\frac{4}{27}$, we have $|D| + |E| < 1$, and thus the roots are inside the unit disc; also the area condition is satisfied. But $[\operatorname{Re}\phi(\omega) - \phi(1)] = -\frac{8}{27}$ and $[\operatorname{Re}\phi(-\bar{\omega}) - \phi(-1)] = \frac{15}{27}$ have opposite signs, so we are unable to conclude as before that there exists θ for which $|A(F_3; \theta)| = 0$. For $p = 0$, $q = -1$, $|A(F_3; \theta)| = L(\frac{5}{4} + \cos 3\theta)$, where $L \neq 0$ (calculation by J. H. Curtiss).

In the last-mentioned (Curtiss) case, $\phi(w) = w + 1/(2w^2)$. The following discussion shows that $A(F_3)$ also cannot be singular for ψ defined by $\psi(w') = \phi(w)$, where $w = r(w' + c)$, for any permissible r and c. At $w' = 1$, $w = r(1 + c)$; therefore r is the radius in the w-scale of the unit circle $|w'| = 1$, and rc is the distance from $w = 0$ to $w' = 0$ in the same scale. The square of the w-distance from the center of $|w'| = 1$ to $w = \omega$ is $(rc + \frac{1}{2})^2 + (3^{1/2}/2)^2$. Therefore the condition for $w = \omega$ to be inside or on $|w'| = 1$ is that $(rc + \frac{1}{2})^2 + (3^{1/2}/2)^2 \leq r$, or

$$\begin{gathered} r^2c^2 + rc - (r^2 - 1) = 0, \\ (-1 - (4r^2 - 3)^{1/2})/2r \leq c \leq (-1 + (4r^2 - 3)^{1/2})/2r. \end{gathered} \tag{11}$$

The idea here is that it may be possible to choose r and c so that the "backward" Fejér points are on a line, to make $A(F_3; \pi)$ singular. Image $z_3 = \phi(-1)$ is at $x_3 = 2r(-1 + c) + 1/(r^2(-1 + c)^2)$. The x-coordinates of images $z_1 = \phi(-\bar{\omega})$ and $z_2 = \phi(-\omega)$ are

$$x_1 = 2r(c + \tfrac{1}{2}) + \frac{1}{r^2} \frac{[(c + \frac{1}{2})^2 - \frac{3}{4}]}{[(c + \frac{1}{2})^2 - \frac{3}{4}]^2 + 3(c + \frac{1}{2})^2}. \tag{12}$$

The following condition may be derived for equality of x_1 and x_3:

(13) $(4\frac{1}{2} - 3r)c^4 + 3(1 - 2r)c^3 - 9rc^2 - 6rc - 3(r - \frac{1}{2}) = 0.$

Condition (11) requires that $c < 1$. For a positive solution for c of (13), we must have $r \leqq \frac{3}{2}$, and for a positive solution with $r \geqq 1$, necessarily $(4\frac{1}{2} - 3r)c^4 > 3r - \frac{3}{2}$,

(14) $$c^4 > \frac{3r - \frac{3}{2}}{4\frac{1}{2} - 3r} \geqq \frac{3 - \frac{3}{2}}{\frac{3}{2}} = 1 .$$

Therefore collinearity of the backward Fejér points is impossible, and matrix $A(F_3; \pi)$ is nonsingular, for all possible choices of r and c. For the case $d = -c$, condition (13) becomes

(15) $(4\frac{1}{2} - 3r)d^4 + 3(2r - 1)d^3 - 9rd^2 + 6rd - 3(r - \frac{1}{2}) = 0.$

The condition for $w = 1$ to be inside of or on $| w' | = 1$ is $(1/r) + d \leqq 1$. Solving (15) for r in terms of d, we obtain

(16) $$r = \frac{4\frac{1}{2}\, d^4 - 3d^3 + 1\frac{1}{2}}{3d^4 - 6d^3 + 9d^2 - 6d + 3}.$$

For $d = 0.1, 0.2, \frac{1}{3}, \frac{1}{2}, r < 1$, contrary to $d > 0$ and $(1/r) + d \leqq 1$. For $d = \frac{2}{3}$, r is about 1, but $(1/r) + d \leqq 1$ requires $r \cong 3$. For $r = 1, d = 1$, so $(1/r) + d = 2$. Apparently in this case also, $A(F_3)$ cannot be singular.

9. A singular Fejér set for $n = 2$. As already noted in §5, the two-parameter family of real-symmetric mappings (5) places the images $\phi(\alpha)$, $\phi(\alpha^2)$ on a line parallel to the imaginary axis in the z-plane, as required for F_5 to be singular by Theorem 2.1. With the choices $c_4 = 0.080$, $a = 0.130$ of parameter values, (5) becomes

(17) $$z = \phi(w) = w + .080 - \frac{.920}{w} + \frac{.210}{w^2} - \frac{.050}{w^3} + \frac{.080}{w^4},$$

and (5**) becomes

(17**) $w^5 + .920w^3 - .420w^2 + .150w - .320 = 0.$

The area condition (5*) is satisfied: $.1485 = 10c_4^2 + 5a^2 < 2c_4(1 + a) = .1808$. The left side of (5) has the approximate factorization

(18) $(w - .667)(w^2 + .692w + .652)(w^2 - .025w + .730),$

which shows, by use of the quadratic coefficient triangle, that the five roots are all well inside $| w | = 1$. The image of $| w | = 1$ under (17) again is an analytic simple closed curve, for which the corresponding Fejér set F_5 is singular.

10. Singular Fejér sets, $n \geqq 3$. Consider the three-parameter family of mappings

$$z = \phi(w) = w - \frac{(1-b)}{w} - \frac{b}{w^{2n}} + a\left[\frac{1}{w^2} - \frac{1}{w^{2n-1}}\right] + c\left[\frac{1}{w^3} - \frac{1}{w^{2n-2}}\right]. \tag{19}$$

The mappings of this family place all of the images of the $(2n+1)$th roots of unity on the imaginary axis, so that according to Theorem 2.1, the matrix $A(F_{2n+1})$ is of rank not greater than $(n+1)$. (It is of rank $(n+1)$ for a, b, c such that the images of the $(2n+1)$th roots are all distinct.) The area condition is

$$(1-b)^2 + 2nb^2 + (2n+1)(a^2+c^2) < 1. \tag{19*}$$

For schlichtness, the roots of

$$w^{2n+1} + (1-b)w^{2n-1} + 2nb + a[-2w^{2n-2} + (2n-1)w] + c[-3w^{2n-3} + (2n-2)w^2] = 0 \tag{19**}$$

must all be in the closed unit disc. Calculations by the author indicate that definitely for $n = 3$, but apparently not for larger n, there is a certain coefficient region of a, b, c compatible with the area condition, such that the roots of (19**) are within the open unit disc, so that there is a corresponding family of analytic curves which have Fejér sets F_7 which are of rank only 4.

The general real-symmetric $\phi(w)$ with $m = 8$ may be written in the form

$$z = \phi(w) = w + \left(\frac{c_1}{w} - \frac{b_1}{w^8}\right) + \left(\frac{c_2}{w^2} - \frac{b_2}{w^7}\right) + \left(\frac{c_3}{w^3} - \frac{b_3}{w^6}\right) + \left(\frac{c_4}{w^4} - \frac{b_4}{w^5}\right). \tag{20}$$

For α the basic ninth root of unity, the real parts of the images under (20) of $\alpha, \alpha^2, \alpha^3 = \omega$ are respectively

$$(c_1 + 1 - b_1)(\alpha + \bar{\alpha})/2 + (c_2 - b_2)(\alpha^2 + \bar{\alpha}^2)/2 + (c_3 - b_3)(\omega + \bar{\omega})/2 + (c_4 - b_4)(\alpha^4 + \bar{\alpha}^4)/2,$$

$$(c_1 + 1 - b_1)(\alpha^2 + \bar{\alpha}^2)/2 + (c_2 - b_2)(\alpha^4 + \bar{\alpha}^4)/2 + (c_3 - b_3)(\omega + \bar{\omega})/2 + (c_4 - b_4)(\alpha + \bar{\alpha})/2,$$

$$(c_1 + 1 - b_1)(\omega + \bar{\omega})/2 + (c_2 - b_2)(\omega + \bar{\omega})/2 + (c_3 - b_3) + (c_4 - b_4)(\omega + \bar{\omega})/2.$$

The requirement that $\phi(\alpha)$, $\phi(\alpha^2)$, $\phi(\omega)$ have the same real part (so that F_9 is singular with rank 8) is satisfied provided that

(21) $$c_1 + 1 - b_1 = c_2 - b_2 = c_3 - b_3 = c_4 - b_4 .$$

Imposing this special requirement, (20) becomes

(22) $$\phi(w) = w + \frac{c_1}{w} + \frac{c_2}{w^2} + \frac{c_3}{w^3} + \frac{c_4}{w^4} - \frac{(1 + c_1 - a)}{w^8} - \frac{(c_2 - a)}{w^7} - \frac{(c_3 - a)}{w^6} - \frac{(c_4 - a)}{w^5}.$$

For schlichtness, the roots of

(22**) $$w^9\phi'(w) = w^9 - c_1w^7 - 2c_2w^6 - 3c_3w^5 - 4c_4w^4 + 5(c_4 - a)w^3 + 6(c_3 - a)w^2 + 7(c_2 - a)w + 8(1 + c_1 - a) = 0$$

must be within the closed unit disc. Put $c_2 = c_4 = 0$, $c_3 = a = (1 + c_1)$, so that (22**) assumes the form

(23**) $$w[w^8 + (1 - a)w^6 - 3aw^4 - 5aw^2 - 7a] = 0.$$

The area condition is

(23*) $$(1 - a)^2 + 3a^2 + 5a^2 + 7a^2 < 1, \quad \text{or} \quad 0 < a < \tfrac{1}{8}.$$

Using Ferrari's solution of the quartic ([5, p. 50]), we see that the roots of (23**) are within the open unit disc provided that

(24) $$[\beta, \gamma] = \left[\frac{1 - a}{2} + \left(\frac{(1 - a)^2}{4} + 3a + y\right)^{1/2}, \quad \frac{1}{2}y + \frac{(4y^2 + 7)^{1/2}}{4}\right]$$

is inside the quadratic coefficient triangle, where y is the positive root of the cubic

(25) $$-y^3 + 3ay^2 - (5a^2 + 23a)y + 7a(1 - a)^2 + 59a^2 = 0.$$

These requirements are satisfied by the choice $a = \frac{1}{16}$. Therefore there exists at least a one-parameter class of analytic simple closed curves, which contain Fejér sets F_9, which are singular because of placement by $z = \phi(w)$ of the six images $\phi(\alpha)$, $\phi(\bar{\alpha})$, $\phi(\alpha^2)$, $\phi(\bar{\alpha}^2)$, $\phi(\omega)$, $\phi(\bar{\omega})$ on the same line parallel to the imaginary axis.

Successive attempts by the author to answer Curtiss' question concerning the possible singularity of $A(F_{2n+1})$ for general n (or for an infinite set of values of n), with satisfaction of conditions (*), (**), yielded only the special cases $n = 2, 3, 4$, and perhaps $n = 5$. The question is still open for larger n.

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