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Reprinted from the AMERICAN MATHEMATICAL MONTHLY
Vol. LXIII, No. 1, January, 1956

## AN EQUIVALENT CONDITION FOR UNIFORM CONVERGENCE

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Given a pointwise convergent sequence of functions from a compact space, it is known that a sufficient condition for uniform convergence is equicontinuity. For the usual type of countable sequence, equicontinuity is also necessary. (See [1], Theorems 24 and 25.) The main purpose of this note is to present another necessary and sufficient condition for uniform convergence of a pointwise convergent sequence of continuous mappings from a compact Hausdorff space. At the end of the note, some generalizations are stated.

One consequence of the following theorem is the Dini theorem on uniform convergence of a monotone sequence. ([1], Theorem 26.)

1. THEOREM. *Let $S$ be a compact space, and $R$ a space with a metric $d\,[\cdot,\cdot]$. Let $\{f_n\}$ be a sequence of continuous mappings on $S$ to $R$, which converges pointwise to a mapping $g$. Then $\{f_n\}$ converges uniformly if and only if* (1) *$g$ is continuous, and* (2) *for arbitrary $\epsilon>0$, there exists an integer $m\geqq 0$ and a $\delta>0$, such that whenever $n>m$ and $d[f_i(s), g(s)]<\delta$, then $d[f_{i+n}(s), g(s)]<\epsilon$.*

*Proof:* The necessity of (1) is well known, and the necessity of (2) is an immediate consequence of the definition of uniform convergence.

To prove sufficiency, choose $\epsilon>0$ and let $m$ and $\delta$ be as given by (2). Since $\{f_n\}$ converges pointwise to $g$, for any $s_0\in S$, there exists an integer $i$ so that $d[f_i(s_0), g(s_0)]<\delta$. Using continuity of $f_i$ and of $g$, there exists a neighborhood of $s_0$, such that for $s$ in this neighborhood, we have $d[f_i(s), g(s)]<\delta$. By compactness of $S$, there exists a finite set of integers $i_1, \cdots, i_p$, such that for every $s\in S$, one of these integers $i$ will satisfy $d[f_i(s), g(s)]<\delta$. Define $k$ to be the largest of the integers $m+i_1, \cdots, m+i_p$. Then for $n>k$ and $s\in S$, there will be an integer $i$ such that $n>m+i$ and $d[f_i(s), g(s)]<\delta$. Consequently, for $n>k$ and all $s\in S$, we have $d[f_n(s), g(s)]<\epsilon$, and the convergence is uniform.

If $g(\cdot)$ of the above theorem is a constant mapping, then it is not required that the topology on $R$ be metric (or have some other uniformity specified). This remark leads to the following corollary.

2. COROLLARY. *Let $Q$ be a closed subset of a topological space $R$, and $\{f_n\}$ any sequence of continuous mappings from a compact space $S$ into $R$. Then for each open set $O$ containing $Q$, there exists an integer $k=k(O)$ independent of $s$ such that $f_n(s)\in O$ for all $s$ and all $n>k$, if and only if* (1) *for each $s\in S$, there exists an integer $k_s(O)$ such that for $n>k_s$, $f_n(s)\in O$, and* (2) *for each neighborhood $N$ of $Q$, there exists a positive integer $m$ and a neighborhood $M$ of $Q$, such that whenever $n>m$ and $f_i(s)\in M$, then $f_{i+n}(s)\in N$.*

*Proof:* Identify the points of $Q$. Then let the mapping $g$ be the constant mapping of $S$ into the point $Q$, and apply the preceding remark.

It might be supposed that if the condition of compactness were removed from Theorem 1 that the theorem would still be true. However, consider the sequence of homomorphisms of the locally compact group of real numbers into the group of complex numbers as given by $f_n(x) = xe^{i/n}$. Although the condition holds here and the convergence is uniform on every bounded set of real numbers, the sequence $\{f_n\}$ does not converge uniformly.

If we consider a sequence $\{f_n\}$ which converges uniformly to $g$ and for which $f_1 = g$ but $f_n \neq g$ for some $n > 1$, we see that the number $m$ of the previous theorem must sometimes be greater than zero no matter how $\delta$ is chosen. The next theorem shows that for the case where the $f_n$'s are iterations of a single mapping, $m$ may always be taken to be zero.

3. THEOREM. *Let $S$ be a compact space with a metric $d[\cdot, \cdot]$, and $f$ a continuous mapping of $S$ into $S$. Suppose that the sequence of iterations of $f$, $\{f^n\}$, converges pointwise to a mapping $g$. Then $\{f^n\}$ converges uniformly if and only if (1) $g$ is continuous, and (2) for arbitrary $\epsilon > 0$, there exists $\delta > 0$, such that if $d[s, g(s)] < \delta$, then for $n = 1, 2, \cdots$, we have $d[f^n(s), g(s)] < \epsilon$.*

*Proof:* Sufficiency of the condition is easily established by applying Theorem 1 with $m = 0$.

To prove necessity, by Theorem 1 there is a $\delta_1 > 0$ and a positive integer $m$, such that for $n > m$ and $d[f^i(s), g(s)] < \delta_1$, we have $d[f^{i+n}(s), g(s)] < \epsilon$. Since $f$ is a continuous function and $f^n(s)$ converges to $g(s)$ for $s \in S$, it follows that for $s \in S$, $f[g(s)] = g(s)$. By continuity of $f^i$ for $i = 1, \cdots, m$, and by compactness of $S$, there is a $\delta$ with $0 < \delta < \delta_1$, such that for $s_1, s_2 \in S$, and $d[s_2, g(s_1)] < \delta$, $f^i(s_2)$ is in the $\epsilon$-neighborhood of $g(s_1)$, for $i = 1, \cdots, m$. Therefore, if $0 < n \leqq m$ and $d[f^i(s), g(s)] < \delta$, we have that $f^{i+n}(s)$ is in the $\epsilon$-neighborhood of $g(s)$. But since $\delta < \delta_1$, this is also true if $n > m$, and so the theorem is proved.

Again, if $g(\cdot)$ is a constant mapping, then the existence of a metric (or other uniformity) on $R$ is not required.

4. COROLLARY. *Let $Q$ be a closed subset of a compact space $S$, and $f$ a continuous mapping of $S$ into $S$. Then $\bigcap_{n=1}^{\infty} f^n(S) \subset Q$ if and only if (1) for each $s \in S$ and each neighborhood $N$ of $Q$, there exists a positive integer $m$ such that $f^n(s) \in N$ for $n > m$, and (2) for each neighborhood $N_1$ of $Q$ there exists a neighborhood $N_2$ of $Q$ such that $f^n(N_2) \subset N_1$ for all positive integers $n$.*

*Proof:* Identify the points of $Q$, and apply the preceding remark and the fact that $S \supset f(S) \supset f^2(S) \supset \cdots$.

Consider the case where $S$ is the interval $0 \leqq s \leqq 1$ and $f(s) = s^2$. Then we have an example where the second part of the condition of Theorem 3 holds, but where $g$ is not continuous. If the point $s = 0$ is identified with the point $s = 1$, then the function $g$ is continuous but the second half of the condition of Theorem 3 does not hold.

5. **Definition.** Let $J$ be a Moore-Smith sequence [2] and let $C$ be a set of conditions on ordered pairs of elements of the sequence. Then let $C$ be said to be a *uniformity* on $J$, if and only if $C$ satisfies the following requirements:

a) Given $j_1$, $j_2$ and $j_3 \in J$ so that $j_3$ follows $j_2$, any condition of $C$ which is satisfied by $(j_1, j_2)$ must be satisfied by $(j_1, j_3)$.

b) Given any element $j \in J$ and condition $c \in C$, there exists an element $j' \in J$ such that $(j, j')$ satisfies $c$.

c) Given $j \in J$, there exists a condition in $C$ such that whenever $(j_1, j_2)$ satisfies the condition, then $j_2$ must follow $j$.

For $J$ the set of positive integers, $C$ may consist of the set of conditions $\{c_m\}$ where each $c_m$ corresponds to the non-negative integer $m$ such that $(j_1, j_2)$ satisfies $c_m$ if and only if $j_2 - j_1 > m$. In this case, Theorem 6, which follows, specializes to Theorem 1.

6. THEOREM. *Let $J$ be a Moore-Smith sequence with a uniformity $C$. Let $S$ be a compact Hausdorff space and $R$ a uniform topological space* [3]. *For each $j \in J$, let $f_j$ be a continuous mapping of $S$ into $R$. Let the sequence of mappings $\{f_j\}$ converge pointwise to a mapping $g$. Then a necessary and sufficient condition that $\{f_j\}$ converges uniformly to $g$ is that $g$ is continuous and that given an index $\alpha$, there exists a condition $c \in C$ and an index $\beta$ such that whenever $f_{j_1}(s) \in \beta[g(s)]$ and $(j_1, j_2)$ satisfies $c$, then $f_{j_2}(s) \in \alpha[g(s)]$.*

*Proof:* The proof of this theorem is entirely analogous to the proof of Theorem 1.

Corollary 2 may also be generalized to Moore-Smith sequences with a uniformity. Theorem 3 may be generalized to non-metric compact spaces.

**References**

1. L. M. Graves, Theory of Functions of Real Variables, New York, McGraw-Hill, 1946, p. 121.

2. E. H. Moore and H. L. Smith, A general theory of limits, Amer. Jour. Math., vol. 44, 1922, pp. 102–121.

3. N. Bourbaki, *Éléments de Mathématique*, Livre III, Chapitre II, *Structures Uniformes*. Actualités Scientifiques et Industrielles, No. 858–1142, Hermann, Paris, 1951, pp. 138–140.