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Math. Annalen 173, 181—190 (1967)

Minkowski Planes

Dedicated to Professor GOTTFRIED KÖTHE on the occasion of his sixtieth Anniversary

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1. Introduction

A Minkowski space is a finite dimensional normed real linear space; in particular a Minkowski plane may be regarded as the set of real coordinate pairs $x=(x_1, x_2)$, $y=(y_1, y_2)$, ..., with a norm having the usual properties

$$\|x\| > 0 \quad \text{if} \quad x \neq \theta = (0,0), \quad \|\theta\| = 0\,;$$
$$\|ax\| = |a| . \|x\| \quad \text{for all real scalars } a\,;$$
$$\|x+y\| \leqq \|x\| + \|y\|\,.$$

With metric defined by $d(x, y) = \|x, y\|$, any Minkowski or Banach space is a (Cauchy) complete metric space. An n-dimensional linear space L_n may be equipped simultaneously with a Euclidean norm or norm derived from an inner product, and with a Minkowski norm $\|x\|$. It is well known that the *unit ball* $B = \{x;\ \|x\| \leqq 1\}$ for the Minkowski norm is a convex body, symmetric with respect to the origin θ, and bounded in the Euclidean norm. Conversely, any such convex body B in L_n is the unit ball for a corresponding Minkowski norm in L_n: for each $y \in L_n$, $y \neq \theta$, there is a unique point x in the boundary of B such that $y = ax$, $a > 0$; define $\|y\| = a$. The boundary C of B, the set of all points of unit norm, will be referred to as the *unit sphere* of the Minkowski space.

A well known necessary and sufficient condition for a norm on a linear space to be derivable from an inner product, and thus for the norm to be Euclidean or Hilbertian, is that the von Neumann-Jordan parallelogram law

$$2\|x\|^2 + 2\|y\|^2 = \|x+y\|^2 + \|x-y\|^2 \tag{1}$$

be satisfied by all pairs of vectors x, y from the linear space. In Euclidean space of dimension $\geqq n$, any set of $(n+1)$ points in general position are the vertices of a non-degenerate n-simplex. Iff the $(n+1)$ points lie in a subspace or flat of $k < n$ dimensions, then they are not in general position, and the simplex which has the points as vertices is degenerate. In this paper a study is made of a simplicial Euclidean characteristic of Minkowski spaces, in particular of planes. It is shown that in case the simplicial Euclidean characteristic of a Minkowski plane is 2, then the norm must be Euclidean. A corollary is the known fact that if it is assumed in advance only that the norm on a linear space satisfies (1) with the $=$ sign replaced by either $\geqq$ or by $\leqq$, then equality must hold and the norm still must be Euclidean. Bounds are obtained for the minimum and maximum possible values of the simplicial Euclidean characteristic, in terms of the linear dimension n of the Minkowski space L_n.

Call a metric space having $(n+1)$ points, in particular any subset of $(n+1)$ points of a metric space with the inherited metric, a *metric* $(n+1)$-tuple; call the $(n+1)$-tuple *Euclidean* in case it is congruent with a subset of $(n+1)$ points of Euclidean space. The affine hull of $(n+1)$ points $v_0, \ldots, v_n$ in a linear space is of dimension at most n. By the Banach-Mazur theorem that $C([0, 1])$ is a universal embedding space for separable metric spaces, any metric $(n+1)$-tuple is isometrically embeddable in a flat of dimension $\leqq n$, and by translation, in a Minkowski subspace of dimension $\leqq n$. This paper addresses itself to a set of questions of the following nature: When is an Euclidean $(n+1)$-tuple in general position isometrically realizable as a subset of a Minkowski space of dimension less than n? How much less than n may the dimension be? Is every Euclidean 4-tuple (tetrahedron) realizable as a subset of the Minkowski plane L_2^∞? Investigate the following notion of $(n+1)$-dimension of a metric space S: the $(n+1)$*-dimension of* S is the smallest integer k, such that each $(n+1)$-tuple of S can be isometrically embedded in a Minkowski space of dimension $\leqq k$. What is the 4-dimension of three-dimensional Euclidean space E_3?

2. A motivating example

The norm (p-norm) for a class of Minkowski spaces L_n^p, $1 \leqq p \leqq \infty$, is defined by $\|x\| = (|x_1|^p + \cdots + |x_n|^p)^{1/p}$, $\|x\| = \max(|x_1|, \ldots, |x_n|)$ for $p = \infty$. The space $L_n^2 = E_n$ of course is Euclidean n-dimensional space; a norm is an inner product norm iff it is a non-singular linear transform of the 2-norm.

Consider the plane L_2^∞, which is congruent with L_2^1 (by a 45° rotation and change of Euclidean scale). The set of the four vertices of the unit sphere of L_2^∞ are at mutual distance 2 from each other. Thus the configuration of the four vertices is congruently embeddable in Euclidean space E_3 as an equilateral tetrahedron or 3-simplex. This suggests the following

Definition. The *simplicial Euclidean characteristic* of a metric space S, $\mathrm{sEc}(S)$, is $\geqq n$ in case there is a subset of $(n+1)$ points of S which is congruently embeddable in E_n as a non-degenerate n-simplex. The $\mathrm{sEc}(S) = n$ in case n is the dimension of the Euclidean simplex of largest dimension which is a congruent image of $(n+1)$ points of S; if $\mathrm{sEc}(S) \geqq n$ for all non-negative integers n, then $\mathrm{sEc}(S) = \infty$.

The simplicial Euclidean characteristic of a one-point space S is 0; of a space S having two or more points, at least 1. In case of an infinite set S, with the discrete metric $d(x, y) = 1$ if $x \neq y$, we have $\mathrm{sEc}(S) = \infty$.

Lemma 1. *Any metric space (or subset) having three points is congruently embeddable in Euclidean space.*

Proof. Denote the distances between the three points by a, b, c; in case $c=a+b$, the three points are embeddable in a Euclidean line, and form a degenerate triangle or 2-simplex. Otherwise we may label the distances so that $c \geqq a$, $c \geqq b$, and by the triangle inequality we have $c<a+b$. In the Cartesian plane, the circle of radius a with center at $(-c/2, 0)$, and the circle of radius b with center at $(c/2, 0)$, meet in two points, such that either point, together with the centers $(-c/2)$, $(c/2)$, are the three vertices of the required Euclidean triangle (non-degenerate).

A subset of four or more points of S need not be congruently Euclidean embeddable: for example in L_2^∞, the points $\delta_1=(1,0)$, $\delta_2=(0,1)$, $-\delta_1$, $-\delta_2$ are at the mutual distances indicated in Figure 1, so both the upper and lower triangles embed as degenerate triangles. The points δ_2 and $-\delta_2$, however, are at positive distance 2 from each other; therefore a Euclidean embedding preserving all the mutual distances of the four points is impossible.

Theorem 1. *For the Minkowski planes* L_2^∞, L_2^p, $p \neq 2$, *we have* $3 \leqq \mathrm{sEc}(L_2^\infty) = \mathrm{sEc}(L_2^1) < 5$, $3 \leqq \mathrm{sEc}(L_2^p)$. *Also* $7 \leqq \mathrm{sEc}(L_3^\infty)$, $5 \leqq \mathrm{sEc}(L_3^1)$.

Proof. As mentioned at the beginning of this section, the configuration of the four vertices of the unit cell of L_2^∞ is congruently embeddable as a non-degenerate 3-simplex; therefore $3 \leqq \mathrm{sEc}(L_2^\infty)$. Similarly the eight vertices of the cubical unit cell of L_3^∞ are all at distance 2 from each other, and likewise for the six vertices of the octahedral unit cell of L_3^1. Therefore the lower bounds for the simplicial Euclidean characteristic of the three-dimensional Minkowski spaces are as stated. The existence of the obvious isometry between L_2^∞ and L_2^1 shows that the simplicial Euclidean characteristics of these two planes are equal.

For $p>2$, two of the triangles of the configuration of the four points $\delta_1+\delta_2$, $\delta_2-\delta_1$, $-\delta_1-\delta_2$, $\delta_1-\delta_2$, having a common diagonal edge, may be Euclidean-embedded by Lemma 1. The non-diagonal edges are of length 2, while the diagonal is of length $2.\,2^{1/p}<2.\,2^{1/2}$. By the Pythagorean theorem, the altitude on the diagonal is of length a given by $a^2=(2^{1/p})^2\ (2^{2-2/p}-1)$. Since $p>2$, the diagonal length $2.\,2^{1/p}$ is less than $2a$. For planar Euclidean embedding of the configuration of four points, it would be necessary that the diagonal length be equal to $2a$. Thus the configuration congruently-Euclidean-embeds as a non-degenerate 3-simplex. For $p<2$, a similar calculation shows that the configuration of the four points δ_1, δ_2, $-\delta_1$, $-\delta_2$ (Figure 1) embeds as a non-degenerate 3-simplex. Therefore for $p \neq 2$, we have $3 \leqq \mathrm{sEc}(L_2^p)$.

Finally, to show that $\mathrm{sEc}(L_2^\infty)<5$, consider the congruent Euclidean embedding of any configuration of six vertices $p_0, p_1, \ldots, p_5$ which is so embeddable. The length of each edge in L_2^∞ is measured either by its horizontal projection, or by its vertical projection; let edges be correspondingly designated as h or v edges. From any vertex, say from p_0, there are five edges; at least three must be alike; by 90° rotation if necessary we may suppose that the three like edges are h. This is indicated in Figure 2. A triangle is degenerate if all three of its edges are alike. For non-degeneracy in E_5 of the embedding of $p_0, \ldots, p_5$, it is necessary that none of the triangles formed by subsets of three of the vertices be degenerate. In Figure 2, for non-degeneracy of triangle $p_0 p_i p_j$, edge $p_i p_j$ must be v; similarly for non-degeneracy of triangle $p_0 p_j p_k$, edge $p_j p_k$ must be v. But then if edge $p_i p_k$ is v, triangle $p_i p_j p_k$ is degenerate; if it is h, triangle $p_0 p_i p_k$

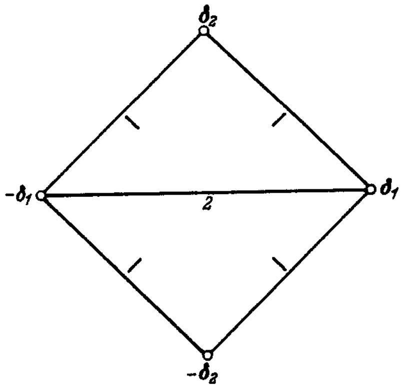

Fig. 1

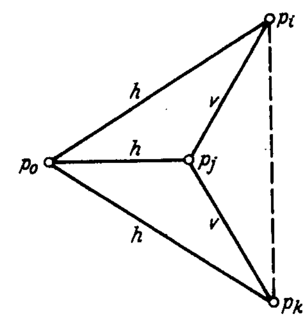

Fig. 2

is degenerate. Therefore a Euclidean-embeddable configuration of six vertices, non-degenerate in its Euclidean embedding as a 5-simplex, is impossible in L_2^∞. Therefore $\mathrm{sEc}(L_2^\infty) < 5$.

3. Classification of triangles and quadruples

It was noted above that a triangle in L_2^∞ is degenerate in its congruent Euclidean embedding if all three of its edges are alike (three h's or three v's). In a triangle with only two like edges, the like edges enclose either an obtuse ($\geqq 90°$) or an acute angle.

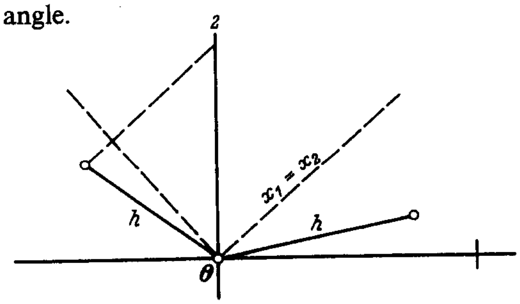

Fig. 3

Lemma 2. *If a triangle in L_2^∞ has two h edges which enclose an obtuse angle, then necessarily the third edge is also h, so that the triangle is degenerate in its congruent Euclidean embedding.*

Proof. In a Minkowski space, distances are invariant under translation; translate the triangle so that the vertex at the obtuse angle is θ. Reflect in the x_1-axis if necessary so that the obtuse angle opens upward, and in the x_2-axis if necessary so that the left h edge is between the negative x_1-axis and the line $x_2 = -x_1$, as indicated in Figure 3. Since the enclosed angle is obtuse, the other

h edge must be below or coincident with the line $x_2 = -x_1$, and the other below or coincident with the line $x_2 = x_1$ (see Figure 3). If the third edge, extending from the vertex of the left h edge, were v, then it could not intersect the line $x_2 = x_1$, and thus it could not meet the vertex of the right h edge. Therefore the third edge must also be h.

Call a configuration of four points a *quadruple*; and in general a configuration of k points a *k-tuple*. Let the polygon which is the outer boundary of the convex hull of a k-tuple be called the *outer convex hull* of the k-tuple. A quadruple of course is degenerate in case its outer convex hull is a degenerate triangle (= configuration of three or four points in an algebraic line).

Lemma 3. *A quadruple of L_2^∞ is degenerate, in its congruent Euclidean embedding, in case its outer convex hull is a triangle.*

Proof. If two like sides of the triangular outer convex hull enclose an obtuse angle, then by Lemma 2 the quadruple contains a degenerate triangle. If two like sides, say both h, enclose an acute angle, then the edge from the common vertex of the two like sides, to the fourth point of the quadruple inside the outer convex hull, also is h; therefore in this case the quadruple is degenerate by the argument used in the proof, given in the preceding section, that $\mathrm{sEc}(L_2^\infty) < 5$. Thus in order that a quadruple be non-degenerate, it is necessary that its outer convex hull be a quadrilateral.

Lemma 4. *A quadruple of L_2^∞ is degenerate, in its congruent Euclidean embedding, in case it has two like outer edges which are adjacent.*

Proof. If the two like edges enclose an obtuse angle, then by Lemma 2 the quadruple contains a degenerate triangle, and hence it is degenerate. If the two like outer edges, say both h, enclose an acute angle, then the inner edge between them also is h; hence the quadruple is degenerate by the argument of the proof that $\mathrm{sEc}(L_2^\infty) < 5$.

By Lemma 4, in order that a quadruple of L_2^∞ be non-degenerate, it is necessary that its outer edges in order be alternately h, v. In case such a quadruple is congruently-Euclidean-embeddable, this condition is also sufficient.

Theorem 2. *For the Minkowski plane L_2^∞, the simplicial Euclidean characteristic is* 3.

Proof. Let $p_0, \ldots, p_4$ be the vertices of any 5-tuple of L_2^∞ which is congruently-Euclidean-embeddable. In case the outer convex hull is a triangle, then by Lemma 3 the 5-tuple contains a degenerate quadruple, so the 5-tuple is degenerate. If the outer convex hull of the 5-tuple is a quadrilateral, then the 4-tuple, formed by the interior point and that triangle of the quadrilateral which contains the interior point (or a triangle of the quadrilateral in case the interior point is on the dividing diagonal), is degenerate by Lemma 3; therefore in this case also the 5-tuple is degenerate. Thus for non-degeneracy of the 5-tuple, it is necessary that the outer convex hull be a pentagon. But the edges of a pentagon cannot be alternately h, v; therefore the 5-tuple contains a 4-tuple which has two like adjacent edges; this 4-tuple is degenerate by Lemma 4.

Thus any congruently-Euclidean-embeddable 5-tuple of L_2^∞ must be degenerate, and Theorem 2 is proved.

Corollary 1. *The simplicial Euclidean characteristic of L_3^∞ does not exceed* 12.

Proof. For any 14-tuple, there are 13 edges from each vertex. In L_3^∞, lengths are measured by their projections on either the 1, 2, or 3-axes. Among the 13 edges, since $4+4+5=13$, there must be at least five which are alike, say 1. For non-degeneracy of a congruent Euclidean embedding of the 14-tuple, all edges involving the five vertices which are the other ends of the five like edges must be 2 or 3. The projection parallel to the 1-axis, into the 2, 3-plane, of the 5-tuple formed by the five vertices, is congruent to a configuration in L_2^∞. But by Theorem 1, the latter configuration is degenerate, so necessarily the 14-tuple ($=13$-simplex) is degenerate. Therefore $12 \geqq \mathrm{sEc}(L_3^\infty)$.

4. Combinatorial corollaries and problems

The result $\mathrm{sEc}(L_2^\infty) \leqq 4$ is purely combinatorial, but $\mathrm{sEc}(L_2^\infty)=3$ is geometric. For Figure 4, for example, shows a configuration of five vertices in which no triangle has three like sides; this verifies that $N(3,3,2)=6$. Since in Ramsey's theorem [3, pp. 38—43], $N(3,5,2)=14$, we have that any configuration of 14 vertices, with three kinds 1, 2, 3 of edges, contains either a triangle with all sides 1, or a 2, 3, 5-tuple (= a 5-tuple with all sides 2 or 3). This is an alternate proof of Corollary 1. Since Ramsey's theorem and $N(3,5,2)=14$ yield also either a triangle with all sides 2, or a 1, 3, 5-tuple, *and* either a triangle with all sides 3, or a 1, 2, 5-tuple, it would seem that it should be possible to reduce the upper bound 12 for $\mathrm{sEc}(L_3^\infty)$ (a triangle with all sides i, or a j, k 5-tuple, for $i=1, 2,$ *or* 3, $j, k \neq i$, would be sufficient for degeneracy of the corresponding simplex). To decide whether n may be less than 14 seems to the writer to be a non-trivial combinatorial problem. Use of the Gleason-Greenwood result [3, p. 42; 2, p. 61], that $N(3,4,2)=9$, yields still another proof of Corollary 1, but again does not improve the bound: In any configuration of n vertices, in which one vertex has nine j or k edges issuing from it, the sub-configuration of the nine end-points of the edges contains either an i triangle or a j, k 4-tuple. The latter together with the vertex constitute the required j, k 5-tuple. In case $n=14$, since $13=4+4+5$, *each* vertex has at least nine j or k edges issuing from it for some j, k. (This property at *one* vertex would be sufficient.) For $n=13$, if five edges from one vertex are alike, we have the desired conclusion; but in a 13-configuration with exactly four edges of each kind 1, 2, 3 from each vertex (such is possible with a spreadeagle pattern from each vertex, but the configuration contains like triangles), we do not have any vertex having nine edges which are only j or k.

For two kinds h, v of edges, define $K(3,q,2)$ to be the least integer, such that it is certain that for each vertex of an n-configuration with $n \geqq K$, there is either a like triangle (= a triangle with all sides alike) containing the vertex, or a like q-tuple not containing the vertex. Define $K(q,3,2)$ to be the same except that "like triangle" and "like q-tuple" are interchanged.

Combinatorial Corollary 2. *We have* $K(3,4,2) \leqq 8$, $K(3,5,2) \leqq 10$, $K(3,q,2) \leqq \leqq 2q$, *in contrast to the Ramsey* $N(3,4,2) = 9$, $N(3,5,2) = 14$. *Also* $K(3,3,2) = N(3,3,2) = 6$, $K(q,3,2) = 7$ *for* $q > 3$. In case of a 6-configuration, $N(3,3,2) = 6$, the conclusion of the Ramsey theorem is only that there is a like triangle; we have the (at first apparently) stronger conclusion that for each vertex there is either a like triangle containing the vertex (with the kind of edge which is most numerous at the vertex), or a like triangle not containing the vertex (with the other kind of edge).

Proof. For $n \geqq (q-1)+q+1$, there are at least q like edges from each vertex, and accordingly there is either a like triangle containing the vertex, or a like q-tuple (with the other kind of edge) which does not contain the vertex.

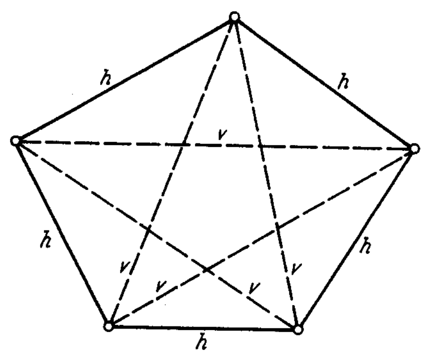

Fig. 4

In a 7-configuration, since $N(3,3,2) = 6$, each sub-configuration of six vertices contains a like triangle. There is a 6-configuration, containing a 5-sub-configuration as in Figure 4, with a vertex (the remaining vertex) for which there is neither a like 4-tuple containing it, nor a like triangle not containing it. Therefore for $q \geqq 4$, we have that $K(q,3,2) = 7$.

Definition. For three kinds 1, 2, 3 of edges, define $V(p,q,2)$ to be the least integer such that an n-configuration with $n \geqq V$ is certain to include, *for each vertex*, either a p-tuple with all sides i containing the vertex, or a j, k q-tuple not containing the vertex. Let $M(p,q,2)$ be the least integer such that if $n \geqq M$, the configuration is certain to include either a like p-tuple, or a j, k q-tuple.

Combinatorial Corollary 3. *We have* $V(3,3,2) = M(3,3,2) = N(3,3,2) = 6$, *and* $M(p,q,2) \leqq V(p,q,2)$. *Also* $V(4,3,2) \leqq 9$, $V(3,4,2) \leqq 11$.

Proof. The first statements are immediate. Given any vertex of any 9-configuration, since $8 = 3+3+2$, there are not more than two of some kind of edge, say i, from the vertex. The remaining six or more edges from the vertex are all j, k; since $N(3,3,2) = 6$, the configuration of the other ends of the non-i edges contains either an i triangle, or a j, k triangle. In the latter case, the configuration of the vertex and the j, k triangle is a j, k 4-tuple, as required.

At each vertex of an 11-configuration, in which there are three kinds 1, 2, 3 of edges, since $3+3+4 = 10$, there must be at least four like edges;

either the other ends of the four edges constitute a j, k 4-tuple, or there is an i triangle at the vertex.

Combinatorial Corollary 4. *We have* $M(p, q, 2) \leqq N(p, q, 2)$, $V(3, 5, 2) \leqq 14$, $V(3, q, 2) \leqq 3q - 1$, $V(5, 3, 2) \leqq 14$.

Proof. If $n \geqq N(p, q, 2)$, then the configuration contains either a 1 p-tuple or a 2, 3 q-tuple, also either a 2 p-tuple or a 1, 3 q-tuple, and either a 3 p-tuple or a 1, 2 q-tuple; while $n = M(p, q, 2)$ requires only either an i p-tuple or a j, k q-tuple. Therefore $M(p, q, 2) \leqq N(p, q, 2)$. The proof of the statement about $V(3, q, 2)$ is similar to the proof above that $V(3, 4, 2) \leqq 11$. The proof that $V(5, 3, 2) \leqq 14$ uses the fact that $N(3, 4, 2) = 9$, and is similar to the proof that $V(4, 3, 2) \leqq 9$.

5. Minkowski planes

With any two vectors x, y of L_2, there are associated two parallelograms: one having vertices θ, x, y, $x + y$; the other having vertices θ, $x + y$, $x - y$, $2x$. By Lemma 1, each of the two triangles having a diagonal of a parallelogram as common side is congruently Euclidean-embeddable. The parallelogram evidently is congruently Euclidean-embeddable if and only if the one diagonal is not longer than the Euclidean length of the other diagonal of the Euclidean parallelogram which is the isometric image of the two triangles. It is also clear that the parallelogram of L_2 embeds as a non-degenerate 3-simplex if and only if the Minkowski length of the other diagonal is shorter than the length of the diagonal of the Euclidean parallelogram. If we have inequality in the parallelogram law (1), in case the $=$ sign in (1) is replaced by $>$, the diagonal is too short in the parallelogram having vertices θ, x, y, $x + y$, so the parallelogram congruently Euclidean-embeds as a non-degenerate 3-simplex. In case the $=$ sign in (1) is replaced by $<$, the diagonal of the first parallelogram is too long for congruent Euclidean-embeddability, but the diagonal of the second parallelogram again is shorter than necessary for embeddability as a planer parallelogram: $4\|x\|^2 + 4\|y\|^2 < 2\|x + y\|^2 + 2\|x - y\|^2$. Therefore the second parallelogram congruently Euclidean-embeds as a non-degenerate 3-simplex. Thus we have

Theorem 3. *In case the norm in a Minkowski plane* L_2 *satisfies* (1) *with* $=$ *replaced by* $\geqq$ *for all* $x, y \in L_2$, *then necessarily the norm is Euclidean and* (1) *holds for all* $x, y \in L_2$. *A similar statement is true in case the norm satisfies* (1) *with* $=$ *replaced by* $\leqq$. *The* sEc(L_2) *is* 2 *iff the norm is Euclidean; otherwise it is at least* 3.

Whether or not the sEc(L_2) may be greater than 3 for a Minkowski plane L_2 remains an open question. Each side of a planar regular pentagon is parallel to a diagonal. Consideration of the possibility of Euclidean embedding of a convex pentagon from a Minkowski plane indicates that in case the pentagon has one side parallel to a diagonal, embeddability of one quadrilateral from the pentagon as a Euclidean 3-simplex (one diagonal shorter than planar Euclidean) implies that another quadrilateral (consisting of two triangles having as common side the side of the pentagon which is parallel to the diagonal) is not Euclidean embeddable at all (because the vertices of the two triangles are nearer than their planar Euclidean distance). Therefore if a Minkowski plane L_2 possibly can have $\mathrm{sEc}(L_2) \geqq 4$, the pentagon which embeds as a Euclidean 4-simplex must have its sides and diagonals in ten different directions. So far as the mutual distances of the vertices of the pentagon are concerned, the norm in the Minkowski plane can be replaced by one which has a symmetric icosagon for its unit sphere.

For $n+1$ points which are the vertices of a planar polygon such that the $(n+1)n/2$ edges and diagonals of the polygon all have different directions, there exists a polygon with $(n+1)n$ vertices, symmetric in the origin, such that the chords with mid-points at the origin have the directions of the edges and diagonals. Therefore there always exists a symmetric semi-norm, i.e., a function satisfying $p(x) \geqq 0$, $p(x)=0$ iff $x=\theta$, $p(\lambda x)=|\lambda|\, p(x)$ for all real scalars λ, having a polygonal unit sphere, to realize any prescribed lengths of the edges and diagonals. Thus the simplicial Euclidean characteristic of a plane with such a semi-norm may be arbitrarily large.

It should be possible to use the Blumenthal determinantal condition [1, pp. 98—100] to show that if the simplicial Euclidean characteristic of a semi-normed plane is sufficiently large, then the semi-norm cannot be a norm; i.e., the star-shaped unit sphere for the semi-norm cannot be convex. For congruent Euclidean embeddability of a pentagon as a non-degenerate 4-simplex, the Blumenthal necessary and sufficient condition is that determinants D_1, D_2, D_3, D_4 have alternating signs, where d_{ij} is the distance from vertex i to vertex j (vertices numbered 0, 1, 2, 3, 4),

$$D_4 = \begin{vmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & d_{01}^2 & d_{02}^2 & d_{03}^2 & d_{04}^2 \\ 1 & d_{10}^2 & 0 & d_{12}^2 & d_{13}^2 & d_{14}^2 \\ 1 & d_{20}^2 & d_{21}^2 & 0 & d_{23}^2 & d_{24}^2 \\ 1 & d_{30}^2 & d_{31}^2 & d_{32}^2 & 0 & d_{34}^2 \\ 1 & d_{40}^2 & d_{41}^2 & d_{42}^2 & d_{43}^2 & 0 \end{vmatrix},$$

and D_3, D_2, D_1 are the similar sub-determinants in the upper left corner of D_4. That sign $D_1 = +$, sign $D_2 = -$, follows from the hypothesis that triangle 012 of the pentagon embeds as a non-degenerate Euclidean triangle.

6. Metric 4-tuples in the max normed plane

Any metric triangle (= any Euclidean triangle) can be realized in L_2^∞, with the longest side in any desired direction. For let the sides of a given non-degenerate triangle be a, b, c, with $a \leqq b \leqq c$; by horizontal-vertical symmetry we may assume that side c is measured by its horizontal projection. Then regardless of the horizontal direction of the side of length c, the opposite vertex may be taken to be the (unique) point of intersection of a horizontal side of a 45°

right triangle with sides a, and of a vertical side of a 45° right triangle with sides b, to realize the given triangle.

Theorem 4. *Not all metric 4-tuples, and in particular not all Euclidean tetrahedra, are realizable in* L_2^∞.

Proof. Let the sides of a given 4-tuple be a, b, c, d, e, f, with f the longest side, or one of the longest sides. Let respectively a, b, f; c, d, f be the sides of the two triangles of the 4-tuple which have the common side f. Then since as remarked above the positions of the opposite vertices are uniquely determined, their possible distance apart, e, also is uniquely determined. Thus L_2^∞ contains only one representative of the interval of all 4-tuples which have adjacent faces a, b, f; c, d, f. Therefore as asserted there are many metric 4-tuples, including Euclidean tetrahedra, which are omitted from L_2^∞.

We now desribe some of the 4-tuples and tetrahedra which are included in L_2^∞. Let vertices $q=(-1, 1)$, $r=(-1, -1)$, $s=(1, -1)$ be fixed, and let a fourth vertex $p=(1+t, 1+t)$ vary with the real parameter t. For $-1 \leqq t < 2 \cdot 3^{-\frac{1}{2}} - 2$, the metric 4-tuple is non-Euclidean; at $t = 2 \cdot 3^{-\frac{1}{2}} - 2$, it is Euclidean planar; and for $2 \cdot 3^{-\frac{1}{2}} - 2 < t < \infty$, it is a symmetrical Euclidean pyramid, with base an equilateral triangle of side 2, the common length of the other three sides being $(2+t)$. For variable fourth vertex $p=(1+t, 1)$, at

$$t = 2((2+3^{\frac{1}{2}})^{\frac{1}{2}} - 1),$$

the 4-tuple is Euclidean planar; for larger t it is non-Euclidean; and for t between -1 and the planar value, it is a Euclidean tetrahedron.

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(Received January 24, 1966)