A PROPERTY OF ALGEBRAIC HOMOGENEITY FOR LINEAR FUNCTION-SPACES

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For different points t_1, t_2 of a set T, a function $x(t)$ *separates* t_1, t_2 in case $x(t_1) \neq x(t_2)$. As in the hypothesis of the well-known Stone-Weierstrass theorem, we say that a linear space $D(T)$ of functions $x(t)$ *separates the points* of T if for each pair t_1, t_2 of points of T, $t_1 \neq t_2$, there exists $x \in D(T)$ such that $x(t_1) \neq x(t_2)$. By defining $x(t) = t(x)$, we may regard any linear space X, for each choice of a total or separating set T of linear functionals t on X ([3], p. 153), as a function-space $D(T)$; for such T, $D(T) = X$ separates the points of T.

If the linear space $C(S, \mathfrak{J})$ of all bounded real continuous functions on an infinite topological space $(S, \mathfrak{J})$ separates the points of S, then it may be seen by use of the theorem on p. 41 of [1] that $\mathfrak{J}$ contains a completely regular topology $\mathfrak{S}$. For each finite subset $\{s_1, \cdots, s_n\}$ of S, by complete regularity there exist functions $x_1, \cdots, x_n$ in $C(S, \mathfrak{S})$ such that $x_i(s_j) = \delta_{ij}$. Therefore arbitrary real values $a_1, \cdots, a_n$ at respectively $s_1, \cdots, s_n$ are fitted by some $x \in C(S, \mathfrak{S})$. (Since $\mathfrak{J} \supset \mathfrak{S}$, we know that $C(S, \mathfrak{J}) \supset C(S, \mathfrak{S})$; by Theorem 3.9 of [1], $C(S, \mathfrak{J}) = C(S, \mathfrak{S})$.) Let us describe this situation by saying that for every n, each set of n different points of S is *n-separated* by the linear space $C(S, \mathfrak{J})$, and that the linear space is *algebraically homogeneous* (a.h.). Clearly any larger linear function-space than an a.h. space is an a.h. space.

A one-dimensional linear space $D(T)$ of real functions may separate the points of T if the cardinality of T is not greater than that of the set of real numbers. It is impossible for an n-dimensional $D(T)$ to m-separate any set of m points of T if $m > n \geqq 1$. Following is an example of an infinite dimensional linear space $D(T)$ which separates the points of T, but which is not a.h. Set T is the set of all integers; $x_1(t)$ assumes a different real value for each $t \in T$, so that it separates the points of T; $x_2(t) = \delta_{2t}, \cdots, x_j(t) = \delta_{jt}, \cdots$; $D(T)$ is the set of all finite linear combinations of $x_1, x_2, \cdots$. The linear space $D(T)$ is not a.h., since for $n \geqq 2$ each set of n negative values of t is not n-separated.

If T is a Hamel basis for a linear space L, then L may be regarded as the space of all functions x on T which are such that the subset of T, where $x(t) \neq 0$ is finite; call such functions *finitely-many valued* functions. Then L is a.h. and evidently for $D(T)$ to be not a.h., it is necessary that $D(T)$ not contain all finitely-many valued functions on T.

THEOREM 1. *Suppose that $D(T)$ is any linear space of real functions on an arbitrary set T. Let N be any n-dimensional linear subspace of $D(T)$. Then there always exist some n points $t_1, \cdots, t_n$ of T which are n-separated by N.*

Proof. By hypothesis, there is a set of n functions $x_1, \cdots, x_n$ which are a basis for N. Denote the n-tuple $(0, \cdots, 0)$ by θ. For any real n-tuple $(c_{11}, \cdots, c_{1n}) \neq \theta$, by the hypotheses that N is n-dimensional and $x_1, \cdots, x_n$ is a basis, there exists t_1 such that $c_{11}x_1(t_1) + \cdots + c_{1n}x_n(t_1) \neq 0$. Choose a second n-tuple $(c_{21}, \cdots, c_{2n}) \neq \theta$, orthogonal to $(x_1(t_1), \cdots, x_n(t_1))$. By the hypotheses, there exists t_2 such that $c_{21}x_1(t_2) + \cdots + c_{2n}x_n(t_2) \neq 0$. Continue this procedure up to and including $(c_{n1}, \cdots, c_{nn})$ and t_n. The matrix product

$$\begin{bmatrix} c_{11} & \cdots & c_{1n} \\ \cdot & & \\ \cdot & & \\ c_{n1} & \cdots & c_{nn} \end{bmatrix} \begin{bmatrix} x_1(t_1) & \cdots & x_1(t_n) \\ \cdot & & \\ \cdot & & \\ x_n(t_1) & \cdots & x_n(t_n) \end{bmatrix} = \begin{bmatrix} d_{11} & d_{12} & \cdots & d_{1n} \\ 0 & d_{22} & \cdots & d_{2n} \\ \cdot & & & \\ 0 & 0 & \cdots & d_{nn} \end{bmatrix}$$

is nonsingular, since by construction $d_{11} \neq 0, d_{22} \neq 0, \cdots, d_{nn} \neq 0$, and the determinant of the triangular matrix has the value $d_{11}d_{22} \cdots d_{nn}$. Therefore each matrix on the left is nonsingular, and for arbitrary values $a_1, \cdots, a_n$, there exist $b_1, \cdots, b_n$ such that $b_1x_1 + \cdots + b_nx_n$ fits the values $a_1, \cdots, a_n$ at $t_1, \cdots, t_n$; i.e., $t_1, \cdots, t_n$ are n-separated by N, as required.

If for each n, each n-dimensional subspace $N \subset D(T)$ n-separates every set of n points of T, then $D(T)$ is an a.h. space; but the converse is not necessarily true. In case $D(T)$ is of finite dimension m, $D(T)$ would be described as being a.h. in case all sets of n points are n-separated by $D(T)$ for each $n \leqq m$. The space of all polynomials of degree $\leqq (m-1)$ in a real variable t is an m-dimensional a.h. space. The three-dimensional space of polynomials of degree $\leqq 1$ in two real variables s, t, however, does not 3-separate all triples of points in the s, t-plane, and thus this space is not a.h.

For any subset R of T, denote by $D_R(T)$ the subspace of all those functions of $D(T)$ which vanish on R. A pair M, N of linear subspaces of $D(T)$ are *algebraic* *complements* in case $D(T) = M \oplus N$; i.e., each $z \in D(T)$ has a unique expression $z = x + y$, $x \in M$, $y \in N$.

THEOREM 2. *For any set R of n points which are n-separated by $D(T)$, $D_R(T)$ and an n-dimensional subspace $N \subset D(T)$ are algebraic complements iff the n points are n-separated by N.*

Proof. Let $R = \{t_1, \cdots, t_n\}$, and suppose that the points of R are n-separated by N. Then if $y_1, \cdots, y_n$ are a basis for N, the sets of values $(1, 0, \cdots, 0)$, $\cdots$, $(0, 0, \cdots, 1)$ may be fitted in succession by linear combinations of $y_1, \cdots, y_n$. Therefore there exist functions $x_1, \cdots, x_n$, which also form a basis for N, such that $x_i(t_j) = \delta_{ij}$. For any $z \in D(T)$, we have $z = z(t_1)x_1 + \cdots + z(t_n)x_n + y = x + y$, $x \in N$, $y \in D_R(T)$, as required.

Suppose conversely that N and $D_R(T)$ are algebraic complements. For any t_i, t_j of R, $i \neq j$, by hypothesis there is a $z \in D(T)$ such that $z(t_i) \neq z(t_j)$. Since $z = x + y$, $y \in D_R(T)$, we must have $x(t_i) \neq x(t_j)$; therefore the points of R are n-separated by N.

THEOREM 3. *Given any space $D(T)$ which separates points, there always exists a completely regular topology $\mathfrak{I}$ for T, such that $D(T) \subset C(T, \mathfrak{I})$. Thus in particular there always is an a.h. function-space which contains $D(T)$.*

Proof. The topology is the weak topology $\mathfrak{I}$([1], pp. 38–39) for T which is induced by $D(T)$. The topology $\mathfrak{I}$ is Hausdorff, because if x separates t_1, t_2, the inverse images of disjoint open intervals respectively containing $x(t_1)$, $x(t_2)$ are disjoint and open. Therefore by Theorem 3.7 on p. 40 of [1], the topology $\mathfrak{I}$ is completely regular.

In [2], Hewitt gives examples of regular Hausdorff spaces $(T, \mathfrak{R})$ which are such that $C(T, \mathfrak{R})$ contains only constants; thus of course $C(T, \mathfrak{R})$ separates no pair of points. For any topological space $(V, \mathfrak{V})$, for $v, v' \in V$, define $v\alpha v'$ in case $x(v) = x(v')$ for all $x \in C(V, \mathfrak{V})$. The relation α is an equivalence relation; if the quotient space of $(V, \mathfrak{V})$ by the α-subsets is $(T, \mathfrak{U})$, then $(T, \mathfrak{U})$ is Hausdorff and $C(T, \mathfrak{U})$ separates the points (α-subsets) of T.

It was remarked at the beginning that for any topology $\mathfrak{U}$ such that $C(T, \mathfrak{U})$ separate points, there exists a completely regular topology $\mathfrak{S}$, $\mathfrak{S} \subset \mathfrak{U}$, with the same continuous functions. If $D(T)$ is a larger linear space of functions on T than $C(T, \mathfrak{U})$, then by Theorem 3 there is a completely regular topology $\mathfrak{I}$ for T which is larger than $\mathfrak{U}$. Thus for any set T and completely regular topology $\mathfrak{S}$ which is smaller than the discrete topology for T, there is an ascending sequence of completely regular topologies up to and including the discrete topology. There is a corresponding ascending sequence of a.h. spaces of bounded functions, up to and including the space $m(T)$ of all bounded functions on T. (By Theorem 3.6 of [1], the spaces of all continuous functions on $(T, \mathfrak{U})$, and of all bounded continuous functions on $(T, \mathfrak{U})$, induce the same completely regular topology.)

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