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1. **Introduction.** This paper contains observations on the nature of the range sets of real-valued, bounded, finitely additive set functions. The notation, definitions, and general setting are the same as in [1].

A finitely many-valued additive function has as range set all sums of subseries of a finite sequence $0, a_1, \cdots, a_k$, see §2. Any bounded additive set function $f(X)$ for which there exists an infinitely disjoint sequence of sets of $\mathfrak{M}$, $X_1, X_2, \cdots$, such that, for all i, $f(X_i) = a_i \neq 0$ and $f(X)$ is completely (that is, countably) additive for sums of X_i's, contains in its range the non-denumerable set of all sums of subseries of the sequence $\{a_i\}$ (the latter is a perfect set). In particular, any non-negative, non-finitely many-valued, additive function has this property and hence has a non-denumerable range. There exists, however, an infinitely many-valued additive function having as range only a countable set of real numbers. Two additional examples of additive functions with non-closed ranges are given (§3).

2. **Finitely many-valued functions.** Let $f(X)$ be an additive set function which assumes only a finite number of different values, including at least one positive and one negative value. Let X' be a set on which f assumes its maximum positive value. Then f can be negative for no subset Y of X', since if this were the case we would have $f(X' - Y) + f(Y) = f(X')$, $f(X' - Y) > f(X')$. Similarly, let X'' be a set for which f assumes its minimum value (negative). Then $f = 0$ for the intersection $X' \cdot X''$, and f obviously is the sum of finitely many-valued non-negative and non-positive disjoint components f' and f'' respectively. If we apply Theorem 4.1 of [1] to f' and f'' separately, it follows that $f(X)$ is the sum of a finite number of two-valued functions $f_1, \cdots, f_k$, having disjoint characteristic sets $X_1, \cdots, X_k$, with $M = X_1 + \cdots + X_k$. That is, for each i, $f_i(X)$ is two-valued for all subsets of X_i in $\mathfrak{M}$, and $f_i(X) = f(XX_i)$ for all $X \in \mathfrak{M}$. Hence we have

Theorem 2.1. *A necessary and sufficient condition that a finite set of numbers S be the range of an additive set function is that there exist a finite series of numbers $0 + r_1 + r_2 + \cdots + r_k$, such that S is the set of sums of all subseries of $0 + r_1 + r_2 + \cdots + r_k$.*

3. **Infinitely many-valued functions.** In this section we consider the ranges of non-negative functions and construct additive functions which have denumerable ranges bounded and unbounded. These constructions are the more interesting in that, as we shall prove, the range of a non-negative additive set function

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must be finite or non-denumerable. As in the last section of [1], we now assume that $\mathfrak{M}$ is closed under denumerable addition.

THEOREM 3.1. *Let* $M = X_1 + \cdots + X_n + \cdots$ *be a decomposition of* M *into mutually exclusive sets of* $\mathfrak{M}$. *Let* f *be a bounded additive function defined on* $\mathfrak{M}$. *Define* $f_i(X) = f(XX_i)$. *Then we have*

$$f(X) = \sum_1^\infty f_i(X) + f_0(X),$$

where f_0 *is defined by this equation and is additive (bounded).*

Proof. We need but show that $\sum_1^\infty f_i(X)$ is convergent for each $X \varepsilon \mathfrak{M}$. Let X be given. Then $\sum f_i(X) = \sum f_{i_1}(X) + \sum f_{i_2}(X)$, where i_1 ranges over all integers such that $f_{i_1}(X) \geq 0$, and i_2 over the integers such that $f_{i_2}(X) < 0$. Now, if $\sup_X |f(X)| = b$, then $\sum f_{i_1}(X) \leq b$ and $|\sum f_{i_2}(X)| \leq b$. Hence $\sum f_i(X)$ is absolutely convergent.

In case $f_0(X)$ as defined in Theorem 3.1 is zero for all X, we say that f is *completely additive across* $X_1, \cdots, X_n, \cdots$.

THEOREM 3.2. *If there exists a sequence of infinitely disjoint sets* $X_1, \cdots, X_n, \cdots$ *of* $\mathfrak{M}$, *such that, for all* i, $f(X_i) \neq 0$, *and such that* f *is completely additive across the sequence, then the range of* f *contains a non-vacuous perfect set, and hence the range is non-denumerable.*

Proof. Let $Y = \sum' X_i$ be any sum of sets (infinite or finite) from the sequence $\{X_n\}$. Then $f(Y) = \sum' f(X_i)$ by the complete additivity property. Hence the range of f includes the non-denumerable set S of sums of all subseries of the absolutely convergent series $0 + \sum_1^\infty f(X_i)$. By a procedure similar to the usual discussion of the Cantor set, it may be shown that in general S is a perfect set. (The Cantor set is the set of sums of subseries of $\sum 2 \cdot 3^{-i}$.)

Now we shall show that every infinitely many-valued non-negative additive function has a non-denumerable range. The following lemmas are to that end.

LEMMA 3.1. *If* f *is bounded, additive, and if for every* X *either* $f(X) = 0$ *or* $|f(X)| > C > 0$, *for a constant* C, *then* f *is finitely many-valued.* (*That is,* 0 *is a limit point of the range of any infinitely many-valued bounded function.*)

Proof. Since f is bounded there exists a maximum positive integer n such that there are n disjoint sets $X_1, \cdots, X_n$ for which $|f(X_i)| > C > 0$. Thus, for any subset X of any X_i, $f(X) = f(X_i)$ or $f(X) = 0$, since, if $0 < f(X) < f(X_i)$, X and $X_i - X$ are two sets for which $f > C$; this contradicts the hypothesis that n is the maximum number of sets with the latter property. Hence f is two-valued on each X_i, and f is finitely many-valued.

LEMMA 3.2. *If f is a bounded non-negative additive function such that there is a sequence of sets of $\mathfrak{M}$, $M = Y_1 \supset Y_2 \supset \cdots$, for which $f(Y_i) > 0$ and $\lim f(Y_i) = 0$, then f is completely additive across $X_i = Y_i - Y_{i+1}$, $i = 1, 2, \cdots$.*

Proof. The sequence $X_1, X_2, \cdots$ is infinitely disjoint, and $\sum_1^\infty f(X_i)$ is finite. Let X be any set. Then

$$f(X) = \sum_1^n f(X_iX) + f(\sum_{n+1}^\infty X_iX).$$

Since f is non-negative, and

$$Y_{n+1} \supset \sum_{n+1}^\infty X_iX, \qquad \lim f(Y_{n+1}) = 0,$$

we have

$$\lim_n f(\sum_{n+1}^\infty X_iX) = 0.$$

Hence $f(X) = \sum_1^\infty f(X_iX)$, and f is completely additive across $X_1, X_2, \cdots$.

THEOREM 3.3. *If f is non-negative and additive there exists a sequence of disjoint sets across which f is completely additive. Thus f is either finitely many-valued or its range contains a non-vacuous perfect set.*

Proof. Suppose f is infinitely many-valued. Choose a set Y_1 on which f is infinitely many-valued and such that $f(Y_1) < f(M)$. Then choose $Y_2 \subset Y_1$ such that $f(Y_2) < f(Y_1)$ and such that f is infinitely many-valued on Y_2. By induction a sequence $M \supset Y_1 \supset Y_2 \cdots$ is chosen where $\{f(Y_i)\}$ is a decreasing sequence of positive numbers (not necessarily approaching 0, however). Now let $Z_i = Y_i - Y_{i+1}$. Then $\{Z_i\}$ is a sequence of disjoint sets and $f(Z_i) > 0$ for all i. Let $X_1 = \sum Z_i$. Consider a partition of $\sum Z_i$ into two infinite sums $\sum'$ and $\sum''$. Let X_2 be the partial sum, say $\sum'$, for which $f(\sum') \leq \frac{1}{2}f(X_1)$. Then partition $\sum'$ into two infinite sums. Proceeding thus by induction we obtain a sequence of sets $X_1 \supset X_2 \supset \cdots$ as in Lemma 3.2, and an application of that lemma completes the proof.

COROLLARY 3.1. *Every number in the range of an infinitely many-valued non-negative additive function f is either isolated or every neighborhood of the number contains a perfect set (non-vacuous) of numbers in the range of f.*

Proof. It is obvious that 0 and $f(M)$ satisfy the conditions of the corollary in view of Theorem 3.3 and Lemma 3.1. Let b be any other number in the range of f which is a limit point of the range: $0 < b < f(M)$. Suppose $f(X_0) = b$. Then $f(X_0X)$ and $f([M - X_0]X)$ are non-negative additive functions, and at least one of them must be infinitely many-valued. Suppose $f(X_0X)$ is infinitely many-valued. Then there is a non-vacuous perfect set in the range in every neighborhood of $f(X_0M) = f(X_0)$. Likewise in case $f([M - X_0]X)$ is infinitely many-valued there is a non-vacuous perfect set of values in the neighborhood of $f([M - X_0]M) = f(M - X_0)$, and hence the same would be true at $f(X_0)$.

In view of the results for the range of a non-negative additive function and the decomposition of bounded additive functions into differences of functions of that type the conclusion of the following theorem is somewhat surprising.

THEOREM 3.4. *There exists a bounded additive function with a denumerable range (assuming well-ordering).*

Proof. Let $\{X_i\}$ be a countable infinitely disjoint sequence of non-empty sets of $\mathfrak{M}$, such that $M = \sum_1^\infty X_i$. Let $\sum c_i$ be any convergent series of positive terms. For each i, let $f_i(X)$ be a $(0, c_i)$-valued additive function defined for subsets of X_i in $\mathfrak{M}$. Any set $X \varepsilon \mathfrak{M}$ may be expressed as $X = \sum_1^\infty X_{i1}$, where $X_{i1} = XX_i$. Let T be the set of integers 1, 2, 3, $\cdots$. Associate with each subset τ of T the class of sets X for which $f_i(X_{i1}) = c_i$ if and only if $i \varepsilon \tau$. Define a set function $\beta(\tau)$ as follows: For finite sets τ, $\beta(\tau) = \sum_{i \varepsilon \tau} c_i$; $\beta(T) = 0$. Extension of β by finite additivity then implies $\beta(\tau) = -\sum_i c_i$, where i is not an element of τ, for sets τ which are complements of finite sets. Choose a well-ordering of the remaining sets τ; $\tau_1, \tau_2, \cdots$. Define $\beta(\tau_1) = 0$. Then by finite additivity the definition of β is extended to include all complements, relative to τ_1 and $T - \tau_1$ respectively, of finite subsets of τ_1 and of $T - \tau_1$. Delete the latter from the well-ordering of sets τ. If τ_λ is the next remaining set, again define $\beta(\tau_\lambda) = 0$, extend β by finite additivity, and delete from the well-ordering all new sets to which the definition of β is extended. Proceeding thus through the well-ordering, β is defined for all subsets τ. Since in order that $+c_i$ appear in the expression for $\beta(\tau)$, it is necessary (not sufficient) that τ contain i, and in order that $-c_i$ appear in the expression for $\beta(\tau)$, it is necessary that τ does not contain i, it is clear that β takes on as values only finite sums $\sum e_i c_i$, where $e_i = +1$ or -1. The latter is a bounded countable set of values. The desired countably many-valued additive $f(X)$ on $\mathfrak{M}$ is defined by $f(X) = \beta(\tau)$, where τ is the set associated with X.

In conclusion we give two further examples of functions with non-closed ranges. For these examples, suppose that the base set M is countable, and let $\mathfrak{M}$ be the field of all subsets of M. Let M be represented by the set of positive integers.

Example 1. Let f_1 be a two-valued additive function which is zero on all finite sets. (For the existence of f_1, see §3 of [1] and references given there.) Let f_2 be the completely additive function determined by $f_2(i) = a_i$ for $i = 1, 2, \cdots$, where all $a_i > 0$ and $\sum a_i = f_1(M) = a$. Define $f = f_1 + f_2$. For any $X \varepsilon \mathfrak{M}$,

$f(X) = f_1(X) + \sum_{i \in X} a_i$. Hence f never assumes the value a although a is a condensation point of values of f.

Example 2. Let f_1 and f_2 be two-valued additive functions which are zero on all finite sets, with $f_1(M) = 1, f_2(M) = b > 0$. Arrange M as an infinite two-way matrix (p_{ij}), where each p_{ij} is a positive integer. For a given set X let X_i be the set of integers j such that $p_{ij} \varepsilon X$ for i fixed. Let X^* be the set of integers i such that $f_1(X_i) = 1$. Define $f_0(X) = f_2(X^*)$. Choose a series $\sum a_i = b$, $a_i > 0$ and define $f_i(X) = a_i f_1(X_i)$. Let $f = f_0 + f_1 + \cdots$. Then $f(X) = f_2(X^*) + \sum_{i \varepsilon X^*} a_i$, and f does not assume the limit b as a value, for whenever $f_2(X^*) = b$ there are an infinity of terms in X^*. This suggests means of constructing more complicated functions.

Reference

1. A. Sobczyk and P. C. Hammer, *A decomposition of additive set functions*, this Journal, vol. 11(1944), pp. 839–846.

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