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Let S be a set which contains at least two points, and denote by $D(S)$ an arbitrary linear space of real-valued functions on S. For any linear subspace $L \subset D(S)$, the *variety* A determined by L is the maximal subset of S on which all functions of L vanish. If L separates points of S, then A either is empty or contains a single point; in the latter case the point may be removed from S without changing the linear space $D(S)$. The *nullity* of a subset $B \subset S$ is the Hamel cardinal dimension of the subspace $D_B(S)$ of all functions which vanish on B. Varieties A which contain B may be determined by various proper subspaces L of $D_B(S)$.

For a subset B of S, if a linear subspace $K \subset D(S)$ has the property that for each function z of $D(S)$, there is a function w of K which coincides with z throughout B, we shall say that K *fits* arbitrary functions of $D(S)$ on B. Call the subset B a *rank-set associated with* K, *relative to* $D(S)$, in case K fits arbitrary functions of $D(S)$ on B, and if furthermore there is no proper linear subspace K_1 of K which has the same property. In this case call K a *rank-space* associated with B, relative to $D(S)$.

THEOREM 1. *Iff a subset* $B \subset S$ *has nullity zero, it is a rank-set associated with the entire space* $D(S)$.

Proof. Suppose that B is a rank-set associated with $D(S)$. Then if the nullity of B were not zero, there would be a function $u \in D(S)$ which vanishes on B and which is not the zero-function θ. Let H be any algebraic complement in $D(S)$ of the one-dimensional subspace of scalar multiples of u; then on B, H fits the values of arbitrary functions $v \in D(S)$, contrary to the hypothesis that there is no proper subspace of $D(S)$ which has this property. Therefore there is no function $u \neq \theta$ in $D(S)$ which vanishes on B; i.e. the nullity of B is zero.

Suppose that B has nullity zero. Then for any function $v \in D(S)$, of course v itself fits the values of v on B. Therefore $D(S)$ is the rank-space associated with B provided that it has no proper subspace which on B fits the values of each $v \in D(S)$. If there were such a subspace, there would be some $v \in D(S)$ and $u \in D(S)$, u not identical with v on S, with $u = v$ on B. But then $(u - v)$ would be a function which vanishes on B, and which on S is different from the zero-function θ, contrary to the hypothesis that B has zero nullity. Thus the rank-space associated with B is the entire space $D(S)$.

As an example of Theorem 1, if S is a closed region of the x, y-plane, bounded by a simple closed curve B, and $D(S)$ is any linear space of functions which are continuous on S and harmonic in the open region bounded by B, then since by the maximum principle B has nullity zero for harmonic functions, the boundary B is a rank-set associated with the entire space $D(S)$. By Theorem 1 of [2], for any n-dimensional space $D(S)$ of harmonic functions on S, there exist rank-sets of n points on B. Further examples and applications to curve-fitting are suggested by reference [1] (brought to the author's attention by J. H. Curtiss).

THEOREM 2. *A subspace K of $D(S)$ is a rank-space, with B as an associated rank-set, iff K is an algebraic complement* (see [2]) *of $D_B(S)$.*

Proof. Suppose that B is a rank-set with K as an associated rank-space. Then for any $z \in D(S)$, by hypothesis there is a function x which fits z on B, so the difference $z - x = y$ is in $D_B(S)$. Assume that $v \in D(S)$ is not the zero-function θ, and $v \in D_B(S) \cap K$. Then a hyperplane K_1 in K which is an algebraic complement of the one-dimensional space $\{cv\}$ fits the functions of $D(S)$ on B as well as K does, contrary to the requirement of the definition of K as a rank-space. Therefore $D_B(S) \cap K = \{\theta\}$, and $D_B(S)$ and K are algebraic complements.

For the converse, if $D_B(S)$ and K are algebraic complements in $D(S)$, then for each $z \in D(S)$, we have uniquely $z = x + y$, $x \in K$, $y \in D_B(S)$. Therefore x and z coincide on B. If each z may be fitted on B by $x_1 \in K_1 \subset K$, then $z - x_1 = y_1$ belongs to $D_B(S)$, and since by hypothesis all of K is required together with $D_B(S)$ to span $D(S)$, by uniqueness it follows that $x_1 = x$, $y_1 = y$, $K_1 = K$. Therefore B is a rank-set with K as an associated rank-space.

COROLLARY 3. *Two rank-spaces K, L have a common associated rank-set A iff they are both algebraic complements of $D_A(S)$. Two rank-sets A, B have a common associated rank-space K iff $D_A(S)$ and $D_B(S)$ are both algebraic complements of K.*

THEOREM 4. *For a rank-space K and each associated rank-set A, there is a unique maximal rank-set $B \supset A$, which has the same associated rank-spaces as A (including K).*

Proof. By Theorem 2, $D_A(S)$ is an algebraic complement of K. Let B be the variety which is determined by the subspace $D_A(S)$. Then $D_B(S) = D_A(S)$. For each s in the complement of B, there is a $y \in D_B(S)$ with $y(s) \neq 0$, so that if $z = x + y$, x cannot fit z at s; therefore B is maximal. By Theorem 2, any complement L of $D_B(S)$ also is a rank-space, with B as a maximal associated rank-set.

THEOREM 5. *Two rank-sets, A and C, associated with the same rank-space K, are contained in the same maximal rank-set B iff the functions of K fit the functions of $D(S)$ on $A \cup C$.* (The alternative possibility is that the functions $z \in D(S)$ are fitted both on A and on C by K, but that there are functions z which require a different function from K for fitting on A than for fitting on C.)

Proof. If $A \cup C \subset B$, where B is a rank-set associated with K, then the functions of K in particular fit the functons of $D(S)$ on $A \cup C$. Conversely, if K fits the functions of $D(S)$ on $A \cup C$, then K is a rank-space associated with $A \cup C$, since no smaller linear subspace will fit the functions of $D(S)$ on either A or C. The variety B which is determined by the subspace $D_{A \cup C}(S)$ is the maximal rank-set containing $A \cup C$.

THEOREM 6. *If A and B are rank-sets, and M is the associated rank-space of $A \cup B$, then M contains the linear span $K+L$ of K and L, where K, L are suitable rank-spaces associated respectively with A, B. It is not always necessary that $M=K+L$.*

Proof. Since M fits the functions of $D(S)$ on $A \cup B$, in particular it fits them on A, and so M contains a K. Similarly M contains an L. Therefore $M \supset K+L$. Abbreviate $D_A(S)$ to D_A, etc. By Theorem 2, M is an algebraic complement of $D_A \cap D_B = D_{A \cup B}$. In case $D_{A \cup B}$ is a proper subspace of D_A and of D_B, and K is a common algebraic complement of D_A and D_B, then $L=K$, $K+L=K$, and $K+L$ is a proper subset of M. In case $L \cap D_A$ complements $D_A \cap D_B$ in D_A, or $K \cap D_B$ complements $D_A \cap D_B$ in D_B, then $M=K+L$.

COROLLARY 7. *Each subset C of S is a rank-set. The associated rank-spaces are the algebraic complements of the subspace $D_C(S)$. A subset C for which the maximal rank-set is all of S has $D(S)$ as associated rank-space.* For example, in case of any linear space of continuous functions on a topological space S, the entire space S is the maximal rank-set which contains each dense subset. (A subset B at which all the functions of $D(S)$ vanish of course has $\{\theta\}$ as its associated rank-space, and conversely all functions of $D(S)$ vanish on each rank-set which is associated with $\{\theta\}$.)

Proof. By Theorem 2, any algebraic complement K of $D_C(S)$ is a rank-space with C as an associated rank-subset. If $\{s \in S: y(s)=0 \text{ for all } y \in D_C(S)\} = S$, then $D_C(S)=D_S(S)=\{\theta\}$, so in this case the algebraic complement is unique; it is $D(S)$. The only other case in which the associated rank-space is unique is when $D_C(S)=D(S)$: for such a rank-set C, the associated rank-space is $\{\theta\}$.

THEOREM 8. *Any finite-dimensional linear subspace K of $D(S)$ is a rank-space. If K is of dimension k, there is at least one associated rank set in S which contains exactly k points.*

Proof. If K is of dimension k, then by Theorem 1 of [2], there is at least one rank-set of k points, which has K as associated rank-space. (Also by Theorem 1 of [2], the various maximal rank-sets associated with K are all varieties determined by subspaces $D_A(S)$, where A is some set $\{s_1, \cdots, s_k\}$ of k different points of S.)

For an infinite dimensional $D(S)$, it need not be true that every linear subspace L of $D(S)$ is a possible rank-space. If L is not a rank-space, then by Theorem 2, L is not the algebraic complement of $D_A(S)$ for any $A \subset S$. Each $D_A(S)$ either contains a nonzero function of L, or is insufficient with L to span $D(S)$, or both. For example, if $D(S)$ is the space of all bounded continuous functions on the open unit interval $S = (0, 1)$, and L is the subspace of all bounded functions on (0, 1) which have an extension to be continuous on the closed interval [0, 1], then although L fits the functions of $D(S)$ on every closed subinterval of (0, 1), it is not the complement of $D_A(S)$ for any $A \subset S$. Therefore L is not a rank-space.

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2. A. Sobczyk, A property of algebraic homogeneity for linear function-spaces, this MONTHLY, the preceding paper.