# The Spectral Basis of a Linear Operator

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# Abstract

The idea of a spectral basis first arises in modular or clock arithmetic but is an even more powerful tool in linear algebra and numerical analysis. The spectral basis of a linear operator, uniquely determined by its minimal polynomial, exhibits the macro structure of a linear operator in terms of the basic building blocks of mutually annihilating idempotents and nilpotents which determine its generalized eigenspaces. These ideas are only part of a much larger program developed by the author in his new book, "New Foundations in Mathematics: The Geometric Concept of Number" (Birkhauser 2013), which uses geometric algebra to present an innovative approach to elementary and advanced mathematics. Starting with linear algebra, geometric algebra offers a simple and robust means of expressing a wide range of ideas in mathematics, physics, and engineering.

Garret Sobczyk

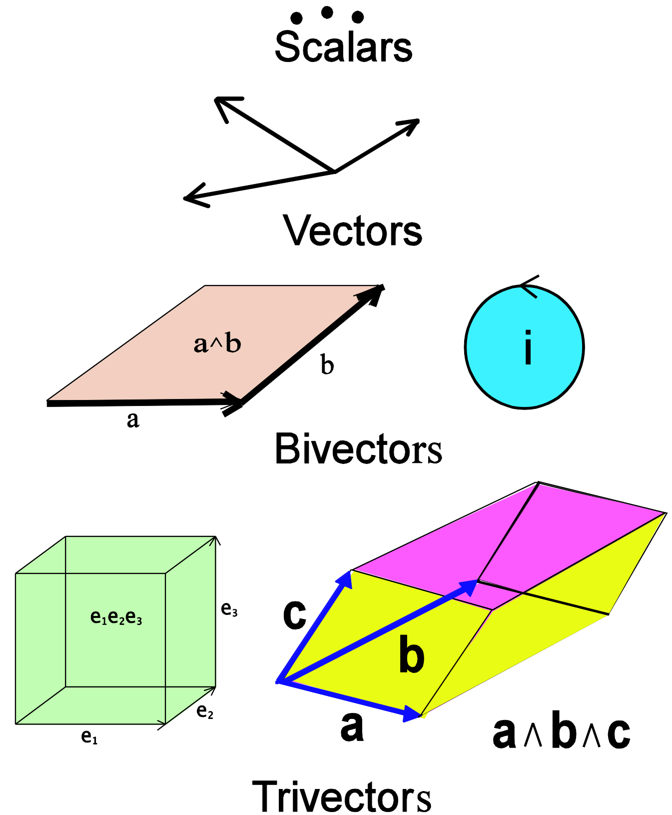
# New Foundations in Mathematics

## The Geometric Concept of Number

Birkhäuser

# What is Geometric Algebra?

Geometric algebra is the completion of the real number system to include new anticommuting square roots of plus and minus one, each such root representing an orthogonal direction in successively higher dimensions.




Geometric numbers of Space

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# Clock Arithmetic

$$n = 12k + r, \text{ where } 0 \leq r < 12$$

$$\Leftrightarrow \quad n = r \mod (12)$$

$12 = 3 \times 2^2$

**Spectral equation**: $s_1 + s_2 = 1$ or

$3(s_1 + s_2) = 3\, s_2 = 3$. This implies that

$9\, s_2 = s_2 = 9$, and $s_1 = 4$. Now define

$q_2 = 2\, s_2 = 6$. **Spectral basis:** $\{ s_1, s_2, q_2 \}$

**idempotents**: $s_1^2 = 16 = 4 \mod 12 = s_1$

$s_2^2 = 81 = 9 \mod 12 = s_2$

**nilpotent**: $q_2^2 = 36 = 0 \mod 12$, $s_1 s_2 = 0 \mod 12$

# Clock Arithmetic: **$12 = 3 \times 2^2$**

**A calculation:** $5s_1 + 5s_2 = 5 \bmod(12)$ **or**

$2s_1 + 1s_2 = 5 \bmod(12)$. **It follows that**

$2^n s_1 + 1^n s_2 = 5^n \bmod(12)$ **for all integers n.**

**$n=-1$ gives $1/5 = 2s_1 + 1s_2 = 5 \bmod(12)$ and**

**$n=100$ gives $5^{100} = s_1 + s_2 = 1 \bmod(12)$.**

# Modular Polynomials and Interpolation mod(h(x))

$$h(x) = (x-1)x^2$$

$$\text{SpectralBasis}:$$

$$s_1 = x^2, \quad s_2 = 1 - x^2$$

$$q_2 = -x^2 + x$$

$$f(x) = \frac{1}{x^2 + 2\sin(x) + 1}$$

$$f(x) = h(x)k(x) + r(x) \text{ where}$$

$$0 \leq \deg(r(x)) < \deg(h(x))$$

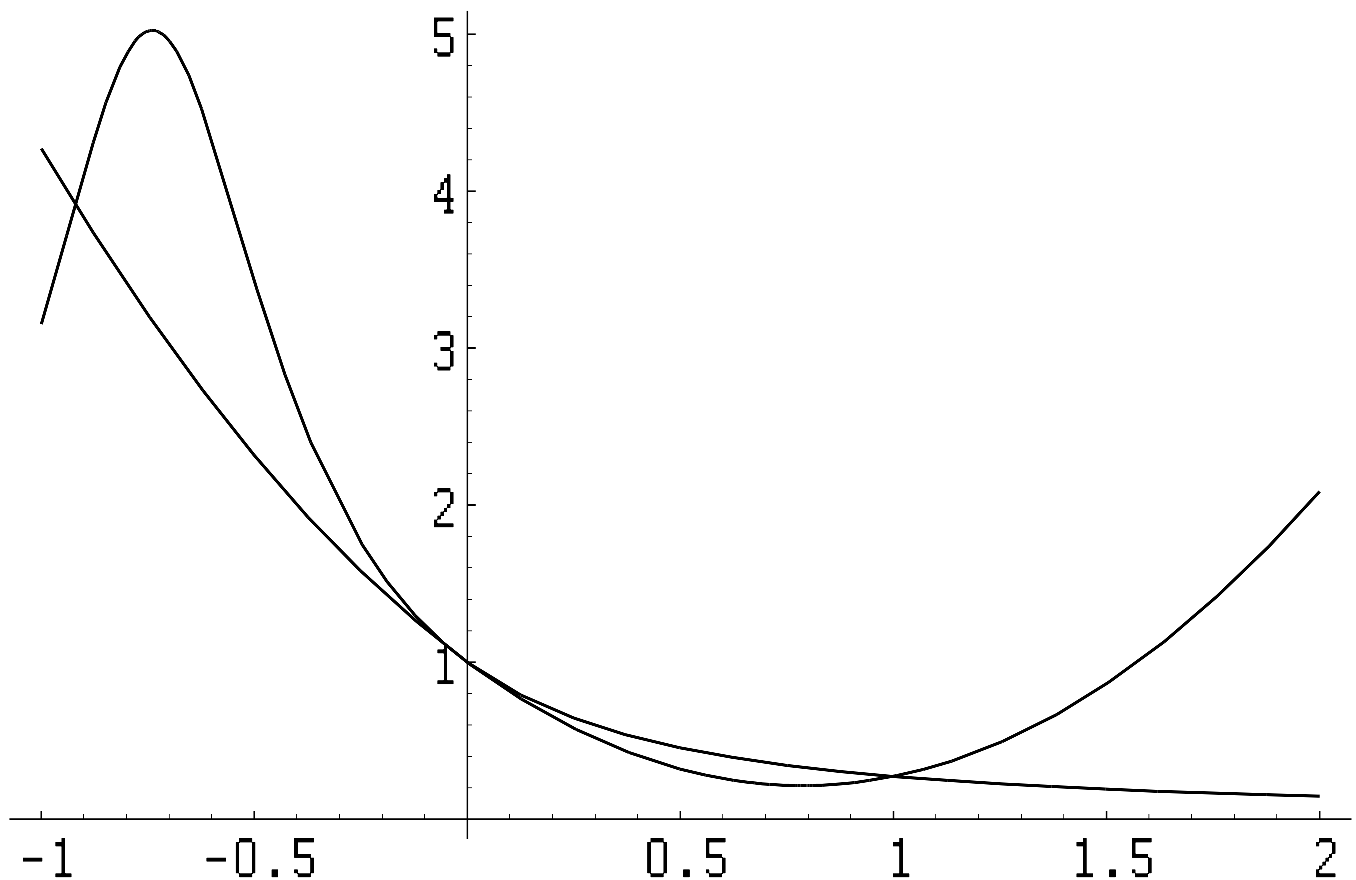
$$\Leftrightarrow \quad f(x) = g(x) \mod (h(x))$$

$$g(x) = f(1)s_1 + f(0)s_2 + f'(0)q_2$$

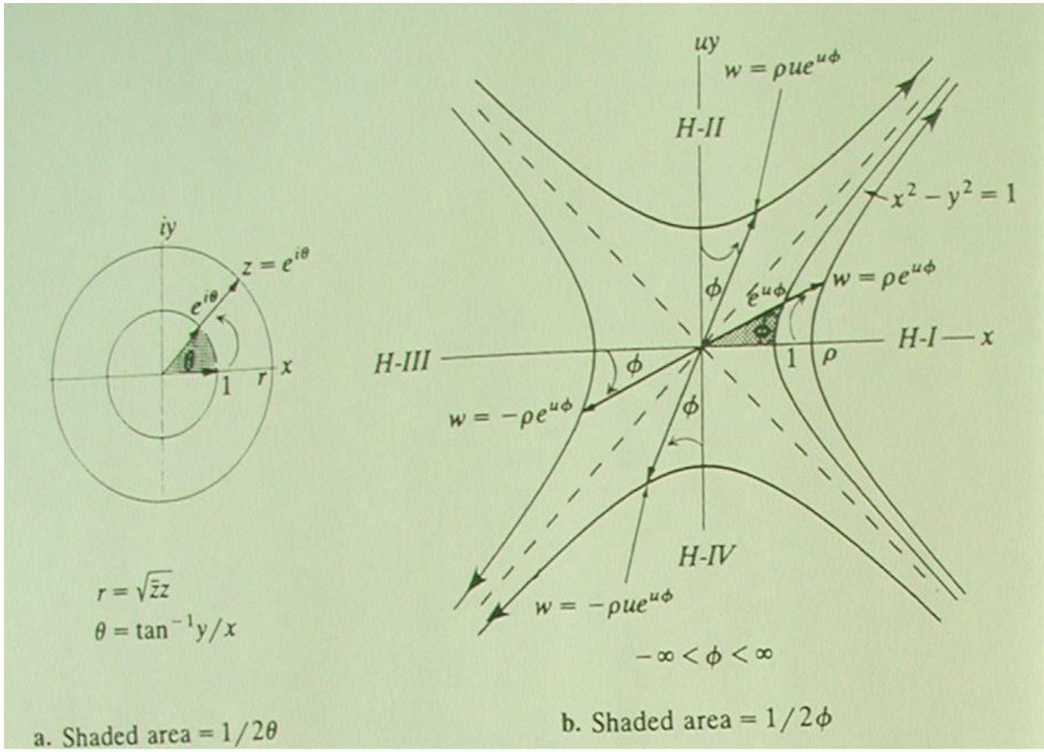
$$= 1 - 2x + x^2\left(1 + \frac{1}{2 + 2\sin(1)}\right).$$

$$f(x) = \frac{1}{x^2 + 2\sin(x) + 1}$$

$$g(x) = 1 - 2x + x^2\left(1 + \frac{1}{2 + 2\sin(1)}\right)$$



# Complex and Hyperpolic Numbers




$$u \notin R$$

$u^2 = 1$

$$z = x + iy = re^{i\theta}, \quad w = x + uy = \rho e^{u\phi}$$

$$i = \sqrt{-1}, \; i^2 = -1, \; u = \sqrt{+1}, \; u^2 = 1$$

# Hyperbolic Numbers

$$w_1 = x_1 + uy_1,\ w_2 = x_2 + uy_2$$

$$w_1 w_2 = x_1 x_2 + y_1 y_2 + (x_1 y_2 + x_2 y_1)u$$

Spectral Basis :

$$w = x + yu = (x+y)u_+ + (x-y)u_-$$

$$\text{where } u_+ = \frac{1}{2}(1+u),\ u_- = \frac{1}{2}(1-u),$$

$$u_+ + u_- = 1,\ u_+ - u_- = u$$

$$u_+^2 = u_+,\ u_-^2 = u_-,\ u_+ u_- = 0.$$

:

# Geometric Numbers $G_2$ of the Plane

Standard Basis of $G_2=\{1, e_1, e_2, e_{12}\}$.

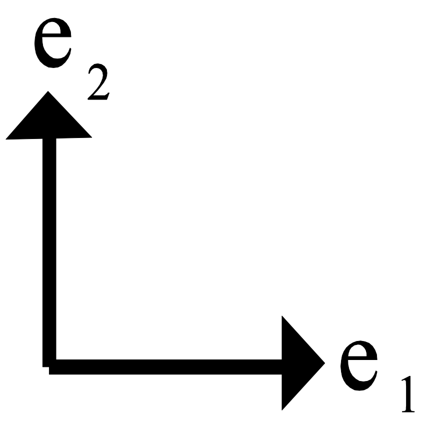
where $i=e_{12}$ is a unit bivector.

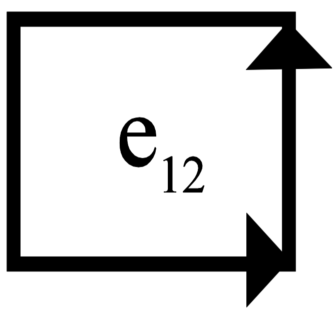
$$e_1^2 = e_2^2 = 1$$

$$e_{12} = e_1 e_2 = -e_2 e_1$$

$$i^2 = (e_1 e_2)^2 = e_1 e_2 e_1 e_2$$

$$= -e_1 e_1 e_2 e_2 = -1$$



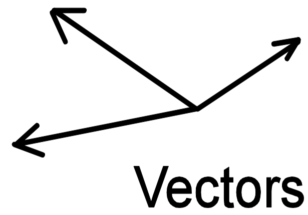
# Basic Identities

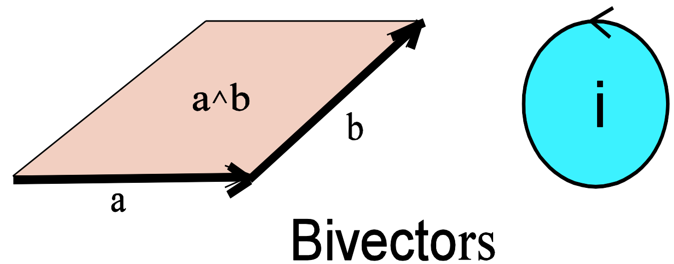
**ab =a·b+a∧b**

**a·b=½(ab+ba)**

**a∧b=½(ab-ba)**

**$a^2$=a·a= |a|$^2$**




$$\mathbf{a} = a_1\mathbf{e_1} + a_2\mathbf{e_2}, \quad \mathbf{b} = b_1\mathbf{e_1} + b_2\mathbf{e_2}$$

$$\mathbf{a}\cdot\mathbf{b} = a_1b_1 + a_2b_2$$

$$\mathbf{a}\wedge\mathbf{b} = (a_1b_2 - a_2b_1)\mathbf{e_{12}}$$

$$\mathbf{ab} = |a||b|e^{i\theta}, \quad i = \mathbf{e_{12}}$$

# Geometric Numbers of 3-Space

$$\mathbf{G}_3 = span\{1, \mathrm{e}_1, \mathrm{e}_2, \mathrm{e}_3, \mathrm{e}_{12}, \mathrm{e}_{13}, \mathrm{e}_{23}, \mathrm{e}_{123}\}$$

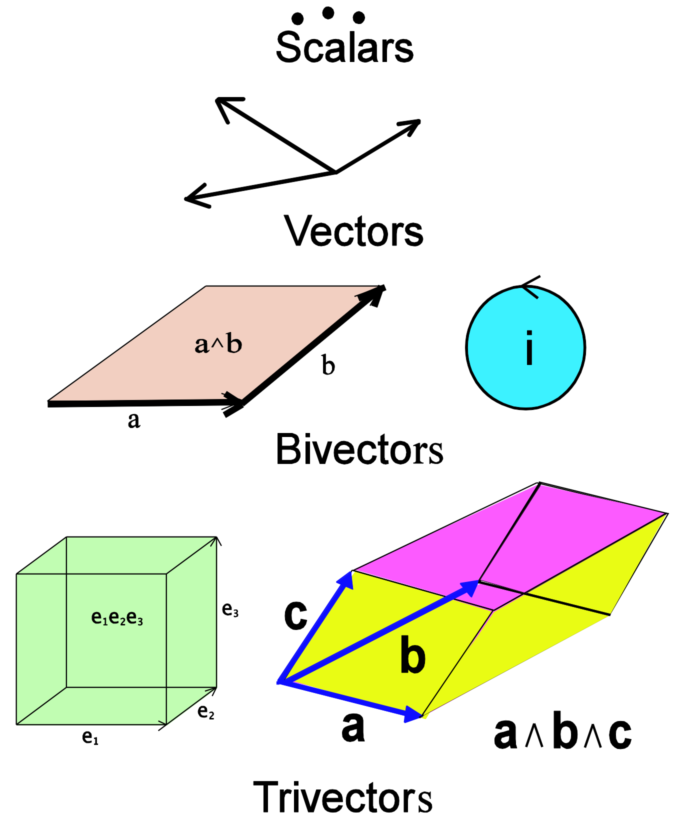
**$a \wedge b = i\, a \times b$**

**$a \wedge b \wedge c = [a \cdot (b \times c)]i$**

**where $i = e_1 e_2 e_3 = e_{123}$**

**$a \cdot (b \wedge c) = (a \cdot b)c - (a \cdot c)b$**

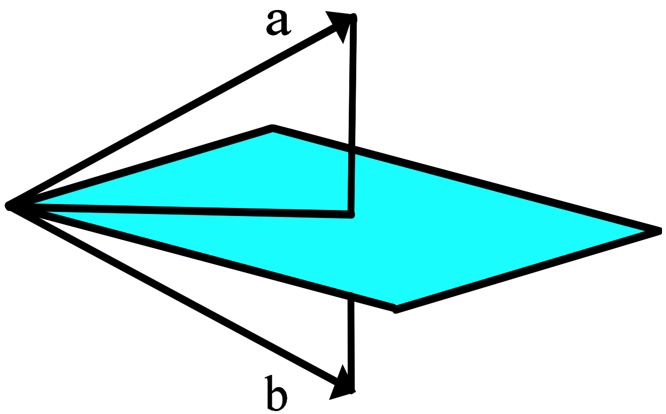
**$= - a \times (b \times c).$**




Geometric numbers of Space

# Reflections L(x) and Rotations R(x)

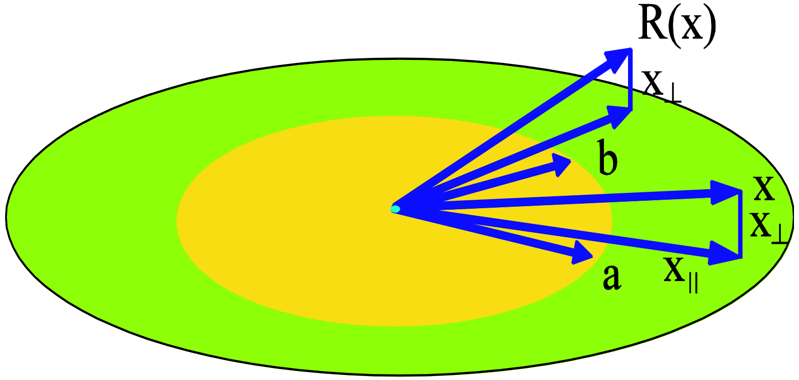
$$L(\mathrm{x}) = -(\mathrm{a}-\mathrm{b})\mathrm{x}(\mathrm{a}-\mathrm{b})^{-1}$$




$$R(\mathbf{x}) = \sqrt{\mathbf{ba}}\;\mathbf{x}\;\sqrt{\mathbf{ab}}$$

where |a|=|b|=1 and

$$\sqrt{\mathbf{ab}} = \pm\mathbf{a}\frac{\mathbf{a}+\mathbf{b}}{|\mathbf{a}+\mathbf{b}|}$$




$$L(\mathrm{a}) = \mathrm{b}, \quad R(\mathrm{a}) = \mathrm{b}$$

# Matrices of the Geometric Algebra $G_2$

Recall that $G_2 = \text{span}\{1, e_1, e_2, e_{12}\}$.

By the spectral basis of $G_2$ we mean

$$\begin{pmatrix} 1 \\ \mathbf{e}_1 \end{pmatrix} u_+ \begin{pmatrix} 1 & \mathbf{e}_1 \end{pmatrix} = \begin{pmatrix} u_+ & \mathbf{e}_1 u_- \\ \mathbf{e}_1 u_+ & u_- \end{pmatrix}$$

where $u_\pm = \frac{1}{2}(1 \pm \mathbf{e}_2)$

are mutually annihiliating idempotents.

Note that $e_1 u_+ = u_- e_1$.

For example, if $[\mathbf{g}] = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad a, b, c, d \in R$

then the element g Ɛ $G_2$ is

$$\mathbf{g} = \begin{pmatrix} 1 & \mathbf{e}_1 \end{pmatrix} u_+ [\mathbf{g}] \begin{pmatrix} 1 \\ \mathbf{e}_1 \end{pmatrix} = a u_+ + b \mathbf{e}_1 u_- + c \mathbf{e}_1 u_+ + d u_-$$

We find that

$$[1] = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, [\mathbf{e}_1] = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \text{and}$$

$$[\mathbf{e}_2] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, [\mathbf{e}_{12}] = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

# Matrices of the Geometric Algebra $G_3$

We can get the complex Pauli matrices from the matrices of $G_2$ by noting that

$e_1 e_2 = i e_3$ or $e_3 = -i e_1 e_2$,

where $i = e_{123}$ is the unit element of volume of $G_3$. We get

$$[\mathbf{e}_1] = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, [\mathbf{e}_2] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \text{ and}$$

$$[\mathbf{e}_3] = -i\mathbf{e}_1\mathbf{e}_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}.$$

# Geometric numbers and determinants

Let $a_1$, $a_2$ , . . . ., $a_n$ be vectors in $R^n$, where

$$\mathbf{a}_i = (\mathbf{e}_1 \quad \mathbf{e}_2 \quad \ldots \quad \mathbf{e}_n) \begin{pmatrix} a_{1i} \\ a_{2i} \\ \cdot\cdot \\ a_{ni} \end{pmatrix} = \sum_{j=1}^{n} a_{ji}\mathbf{e}_i$$

Then

$$\mathbf{a}_1 \wedge \mathbf{a_2} \wedge \cdots \wedge \mathbf{a}_n = \frac{1}{n!} \sum_{\pi \in \Pi} sgn(\pi)\mathbf{a}_{\pi(1)} \cdots \mathbf{a}_{\pi(k)}$$

$$= \det \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \mathbf{e}_{12\cdots n}$$

# Spectral Decomposition

Let $\mathbf{X} = \begin{pmatrix} 9 & -4 & 2 \\ 16 & -7 & 2 \\ -15 & 7 & -1 \end{pmatrix}$ with the characteristic polynomial

$\varphi(x)=(x-1)x^2$. Recall that the spectral basis for this polynomial was

$$s_1 = x^2, \; s_2 = 1 - x^2, \; q_2 = -x^2 + x.$$

Replacing x by the matrix X, and 1 by the identity 3x3 matrix gives

$$\mathbf{S}_1 = \begin{pmatrix} 2 & -1 & 0 \\ 2 & -1 & 0 \\ -8 & 4 & 0 \end{pmatrix}, \mathbf{S}_2 = \begin{pmatrix} -1 & 1 & 0 \\ -2 & 2 & 0 \\ 8 & -4 & 1 \end{pmatrix}, \mathbf{Q}_2 = \begin{pmatrix} 7 & -3 & 1 \\ 14 & -6 & 2 \\ -7 & 3 & -1 \end{pmatrix}.$$

It follows that the spectral equation for X is

$X=1\ S_1 + 0\ S_2 + Q_2$,

with the eigenvectors

$$\mathbf{v}_1 = \mathbf{S}_1 \begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}2\\2\\-8\end{pmatrix}, \mathbf{v}_2 = \mathbf{S}_2 \begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}-1\\-2\\8\end{pmatrix}, \mathbf{v}_3 = \mathbf{Q}_2 \begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}7\\14\\-7\end{pmatrix}.$$

We now obtain the Jordan Normal Form for X

$$\begin{pmatrix}\mathbf{v}_1 & \mathbf{v_2} & \mathbf{v}_3\end{pmatrix}^{-1} \mathbf{X} \begin{pmatrix}\mathbf{v}_1 & \mathbf{v_2} & \mathbf{v}_3\end{pmatrix} = \begin{pmatrix}1 & 0 & 0\\0 & 0 & 1\\0 & 0 & 0\end{pmatrix}$$

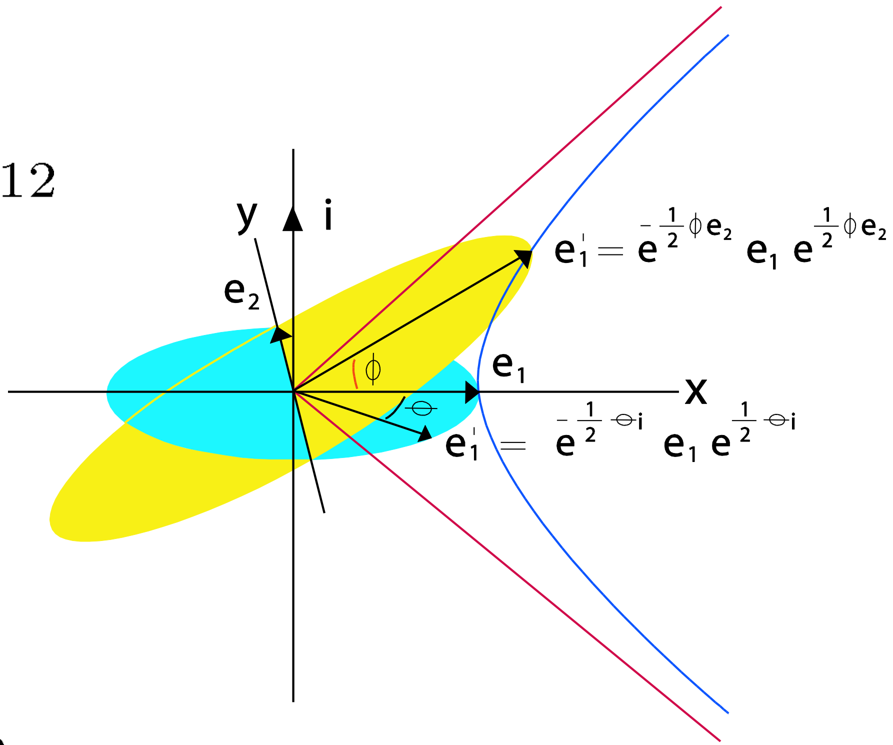
$$f(\mathbf{X}) = \begin{pmatrix} -17+2\left(1+\frac{1}{2+2\sin(1)}\right) & 7-\frac{1}{2+2\sin(1)} & -2\\ -32+2\left(1+\frac{1}{2+2\sin(1)}\right) & 14-\frac{1}{2+2\sin(1)} & -4\\ 30-8\left(1+\frac{1}{2+2\sin(1)}\right) & -14+4\left(1+\frac{1}{2+2\sin(1)}\right) & 3 \end{pmatrix}$$

# Splitting Space and Time

The ordinary rotation

$$R(\mathbf{x}) = e^{-\frac{1}{2}\theta \mathbf{e}_{12}} \mathbf{x} e^{\frac{1}{2}\theta \mathbf{e}_{12}}$$

is in the blue plane of the bivector i=$e_{12}$. The blue plane is boosted into the yellow plane by




$$B(\mathbf{x}) = e^{-\frac{1}{2}\phi \mathbf{e}_{2}} \mathbf{x} e^{\frac{1}{2}\phi \mathbf{e}_{2}}$$

with the velocity v/c = Tanh $\phi$.
The light cone is shown in red.

# Minkowski Space $R^{1,3}$

$$\mathbf{R}^{1,3} = span\{\gamma_0, \gamma_1, \gamma_2, \gamma_3\}$$

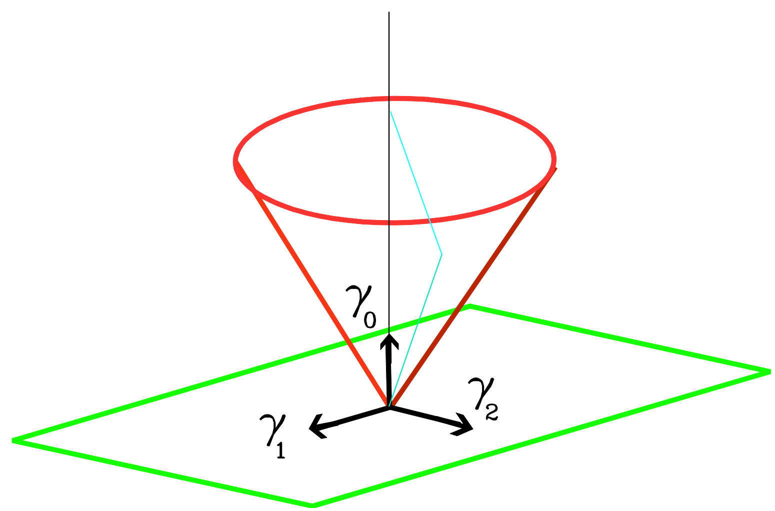
$$\gamma_1^2 = \gamma_2^2 = \gamma_3^2 = -1, \gamma_0^2 = 1$$

$$\gamma_\mu \gamma_\nu = -\gamma_\nu \gamma_\mu \text{ for } \mu \neq \nu,$$

$$0 \leq \mu, \nu \leq 3$$

$g_0$ is timelike,

$g_1$ $g_2$ $g_3$ spacelike



# Spacetime Algebra $G_{1,3}$

We start with

$$\mathbf{G_3} = span\{1, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_{12}, \mathbf{e}_{13}, \mathbf{e}_{23}, \mathbf{e}_{123}\}$$

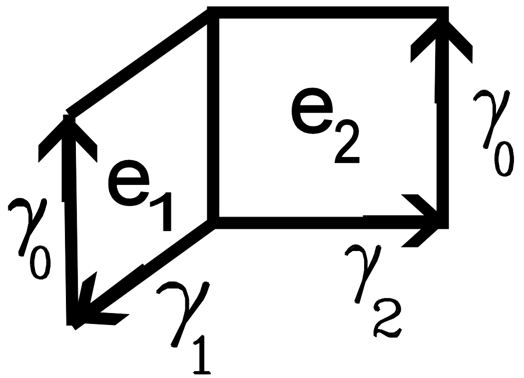
We factor $e_1$, $e_2$, $e_3$ into Dirac bivectors,

$$\mathbf{e}_1 = \gamma_1\gamma_0, \mathbf{e}_2 = \gamma_2\gamma_0, \mathbf{e}_3 = \gamma_3\gamma_0,$$

where

$$\gamma_1^2 = \gamma_2^2 = \gamma_3^2 = -1, \gamma_0^2 = 1$$

$$\gamma_\mu\gamma_\nu = -\gamma_\nu\gamma_\mu \text{ for } \mu \neq \nu.$$




$$\mathbf{G}_{1,3} = span\{1, \gamma_\mu, \gamma_{\mu\nu}, \gamma_{\mu\nu\omega}, \gamma_{0123}\}$$

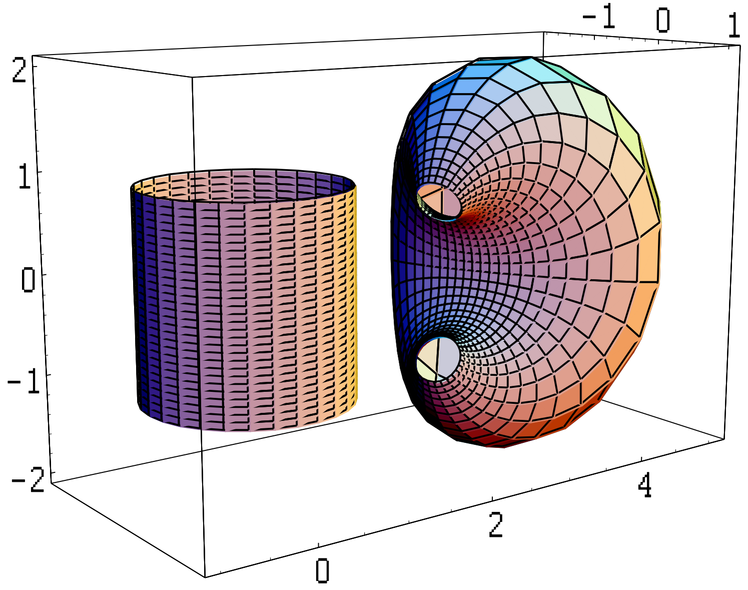
# Conformal Mappings

Conformal mapping the unit cylinder onto the figure shown.

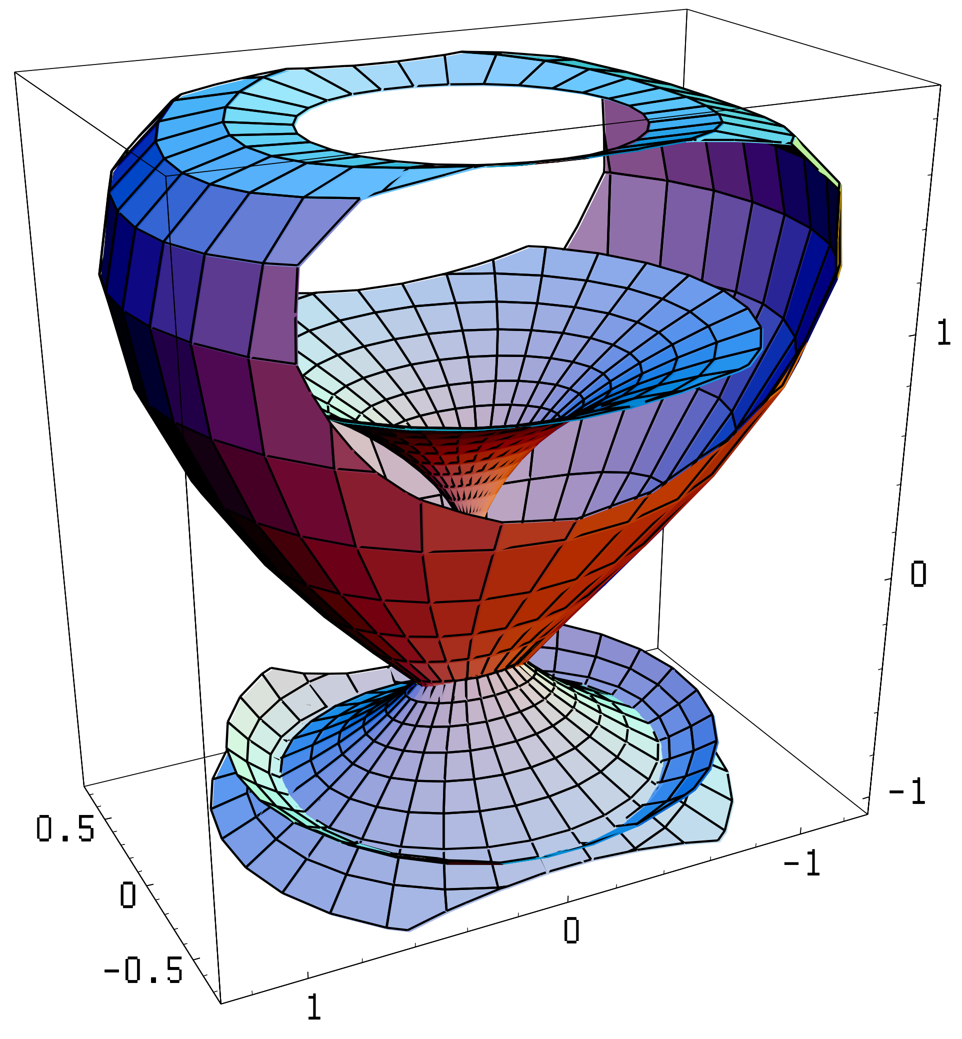
$$f(\mathbf{x}) = \frac{\mathbf{x}-\mathbf{x}^2\mathbf{c}}{1-2\mathbf{c}\cdot\mathbf{x}-\mathbf{c}^2\mathbf{x}^2}$$

$$= \mathbf{x}(1-\mathbf{c}\mathbf{x})^{-1}$$

$$\mathbf{c} = (-1/4, 1/2, 0)$$

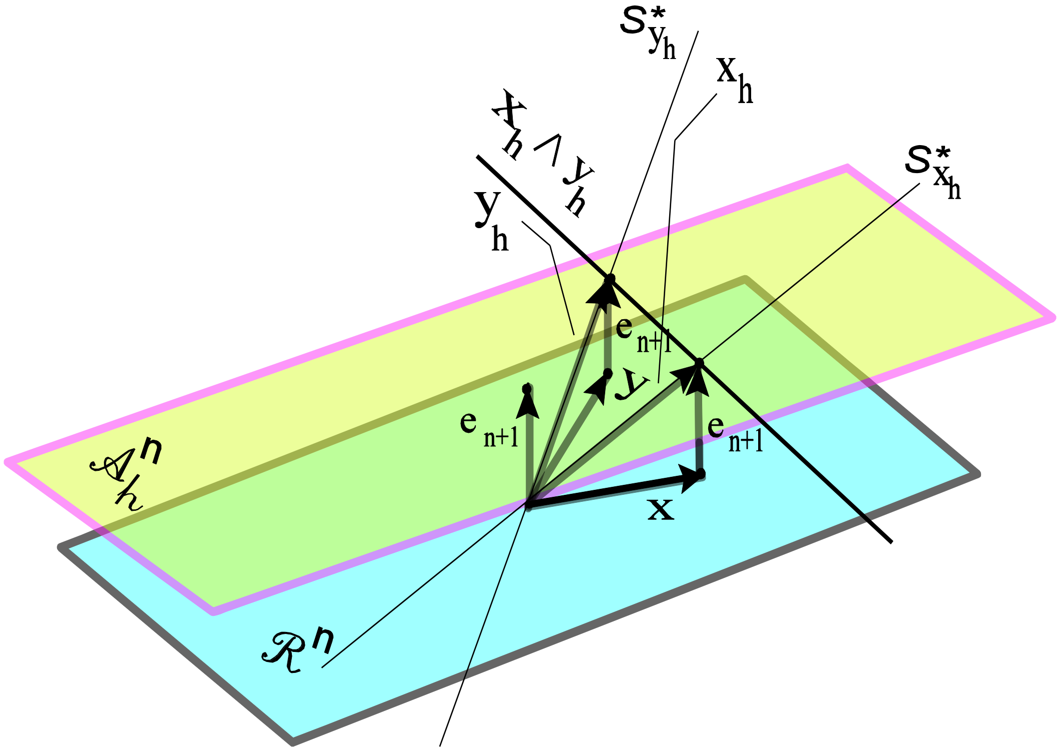


A more exotic conformal mapping of the hyperboloid like figure into the figure surrounding it.



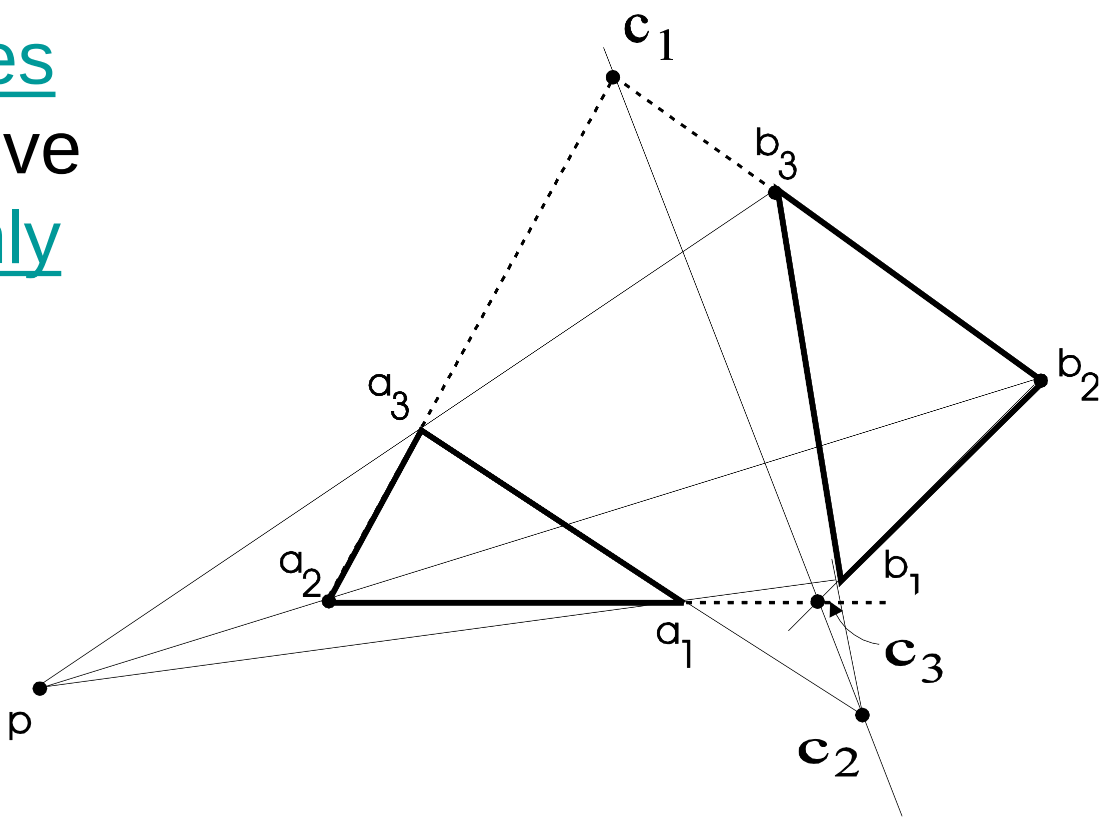
# Non-Euclidean and Projective Geometries

The affine plane. Each point x in $R^n$ determines a unique point $x_h$ in the affine plane.



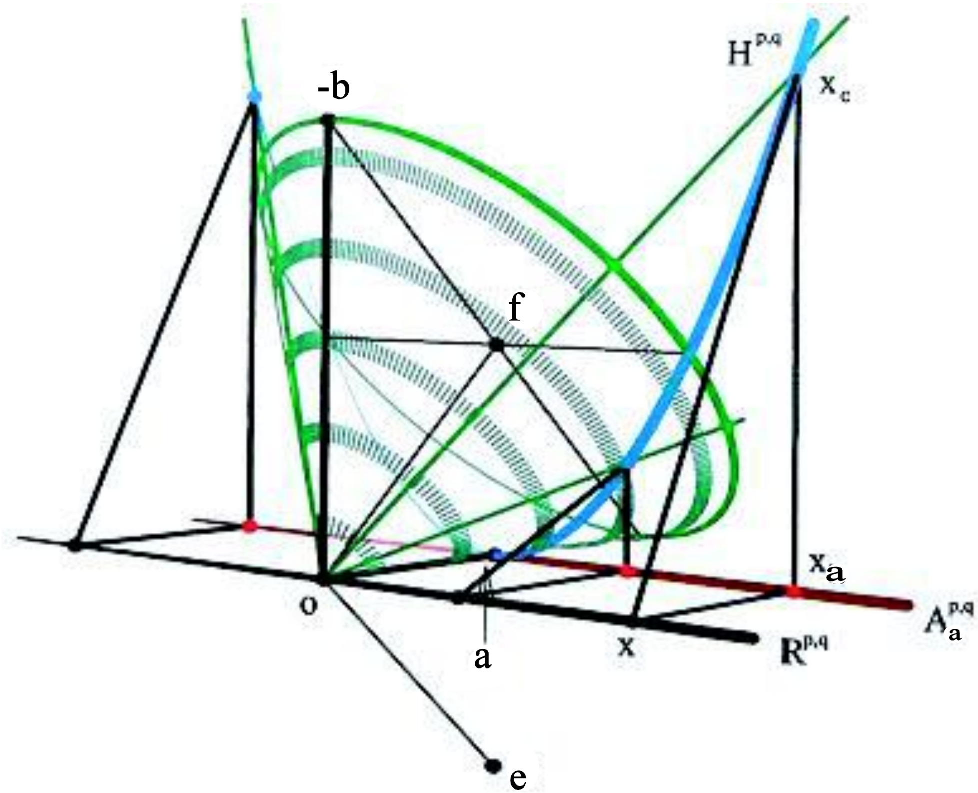
# Desargue's Configuration

Thm: Two triangles are in perspective *axially* if and only if they are in perspective *centrally*.



# The Horosphere

Any conformal transformation can be represented by an orthogonal transformation on the horosphere.



| | $G$ | $[G]$ | $(ax+b)(cx+d)^{-1}$ |
|---|---|---|---|
| Translation | $e^{\frac{1}{2}yb}$ | $\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}$ | $x+y$ |
| Inversion | $e$ | $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ | $\frac{1}{x}$ |
| Dilation | $e^{\frac{1}{2}\phi u}$ | $\begin{pmatrix} e^{\frac{1}{2}\phi} & 0 \\ 0 & e^{-\frac{1}{2}\phi} \end{pmatrix}$ | $e^{\phi}x$ |
| Reflection | $y$ | $\begin{pmatrix} & 0 \\ 0 & -y \end{pmatrix}$ | $-yxy^{-1}$ |
| Transversion | $e^{ca}$ | $\begin{pmatrix} 1 & 0 \\ - & 1 \end{pmatrix}$ | $x(1-cx)^{-1}$ |

# Lie Algebras and Lie Groups

Let $G_{n,n}$ be the $2^{2n}$-dimensional geometric algebra with neutral signature. The Witte basis consists of two dual null cones:

$$\mathcal{A}^n = (\mathbf{a})_{(n)} = \{\mathbf{a}_i \mid 1 \le i \le n\} \text{ and}$$

$$\mathcal{B}^n = (\mathbf{b})_{(n)} = \{\mathbf{b}_i \mid 1 \le i \le n\}, \text{ where}$$

$$\mathbf{a}_i \cdot \mathbf{a}_j = \mathbf{b}_i \cdot \mathbf{b}_j = 0, \text{ and}$$

$$\mathbf{a}_i \cdot \mathbf{b}_j = \delta_{ij} \text{ for all } i, j.$$

We now construct the matrix of bivectors

$$(\mathbf{a})^T_{(n)} \wedge (\mathbf{b})_{(n)} = \begin{pmatrix} \mathbf{a}_1 \wedge \mathbf{b}_1 & \mathbf{a}_1 \wedge \mathbf{b}_2 & \dots & \mathbf{a}_1 \wedge \mathbf{b}_n \\ \mathbf{a}_2 \wedge \mathbf{b}_1 & \mathbf{a}_2 \wedge \mathbf{b}_2 & \dots & \mathbf{a}_2 \wedge \mathbf{b}_n \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \mathbf{a}_n \wedge \mathbf{b}_1 & \mathbf{a}_n \wedge \mathbf{b}_2 & \dots & \mathbf{a}_n \wedge \mathbf{b}_n \end{pmatrix}.$$

These bivectors are the generators of the general linear Lie algebra $gl_n$,

$$gl_n = \{\mathbf{F} = (\mathbf{a})_{(n)} \wedge [f](\mathbf{b})^T_{(n)} \mid [f] \text{ is a real } (\mathrm{n} \times \mathrm{n}) \text{ matrix}\},$$

with the Lie bracket product

$$\mathbf{F}_1 \otimes \mathbf{F}_2 = \frac{1}{2}\Big(\mathbf{F}_1\mathbf{F}_2 - \mathbf{F}_1\mathbf{F}_2\Big) = (\mathbf{a})_{(\mathbf{n})}\big([\mathbf{f_1}][\mathbf{f_2}] - [\mathbf{f_2}][\mathbf{f_1}]\big) \wedge (\mathbf{b})^{\mathbf{T}}_{(\mathbf{n})}.$$

Each bivector F generates a linear transformation f, $f : \mathcal{A}^n \longrightarrow \mathcal{A}^n$ defined by

$$f(\mathbf{x}) = f((\mathbf{a})_{(n)}(x)^T_{(n)}) = (\mathbf{a})_{(n)}[f](x)^T_{(n)} = \mathbf{F} \cdot \mathbf{x}$$

$$\text{for } \mathbf{x} = (\mathbf{a})_{(n)}(x)^T_{(n)} = \sum_{i=1}^{n} x_i \mathbf{a}_i$$

# General Linear Group

The general linear group $GL_n$ is obtained from the Lie algebra $gl_n$ by exponentiation. We have

$$GL_n = \{ G=e^F \mid F \in gl_n \}.$$

Consider now the one parameter subgroups defined for each $G \in gl_n$ by

$$g_t(x)=e^{½tF}\, x\, e^{-½tF}$$

where $x=\sum x_i a_i$ and $t \in R$. Differentiating gives

$$g_t'(\mathbf{x}) = \frac{d}{dt} g_t(\mathbf{x}) = e^{\frac{1}{2}t\mathbf{F}} F \cdot \mathbf{x} e^{\frac{-1}{2}t\mathbf{F}} = g_t(F \cdot \mathbf{x})$$

It follows that $g_t(\mathbf{x}) = e^{tf}\mathbf{x},\ where$ $f(\mathbf{x}) = \mathbf{F} \cdot \mathbf{x}.$

# Conclusions

- The spectral basis of idempotents and nilpotents define both the eigenvalues and eigenspaces of a linear operator.
- The spectral basis also has important applications in number theory, numerical analysis and advanced mathematics.
- Geometric algebra is built upon the geometric concept of number and offers new tools for the study of linear algebra and projective geometry, and provides a unified approach to diverse areas of mathematics, physics and the engineering sciences.
- I hope my selection of topics has been sufficiently broad to show that geometric algebra and the Geometric Concept of Number provides a New Foundation for Mathematics.

# Selected References


W.K. Clifford, *Applications of Grassmann's extensive algebra*, Amer. J. of Math. 1 (1878), 350-358.

P. J. Davis, *Interpolation and Approximation*, Dover Publications, New York, 1975.

T.F. Havel, *GEOMETRIC ALGEBRA: Parallel Processing for the Mind* (Nuclear Engineering) 2002. http://www.garretstar.com/secciones/

D. Hestenes, *New Foundations for Classical Mechanics, 2nd Ed.,* Kluwer 1999.

D. Hestenes and G. Sobczyk. *Clifford Algebra to Geometric Calculus: A Unified Language for Mathematics and Physics*, 2nd edition, Kluwer 1992.

G. Sobczyk, *The missing spectral basis in algebra and number theory*, The American Mathematical Monthly 108 April 2001, pp. 336-346.

P. Lounesto, *Clifford Algebras and Spinors, 2nd Edition*. Cambridge University Press, Cambridge, 2001.

G. Sobczyk, *Hyperbolic Number Plane*, The College Mathematics Journal, 26:4 (1995) 268-280.

G. Sobczyk, *The Generalized Spectral Decomposition of a Linear Operator*, The College Mathematics Journal, 28:1 (1997) 27-38.

G. Sobczyk, Spacetime vector analysis, Physics Letters, 84A, 45 (1981).

J. Pozo and G. Sobczyk. *Geometric Algebra in Linear Algebra and Geometry*}. Acta Applicandae Mathematicae, 71: 207--244, 2002.


Note: Copies of many of my papers and talks can be found on my website: http://www.garretstar.com