Geometric Algebra in Linear Algebra and Geometry

José María Pozo
Departament de Física Fonamental
Universitat de Barcelona
Diagonal 647, E-08028 Barcelona, Spain
jpozo@ffn.ub.es

Garret Sobczyk
Departamento de Fisica y Matematicas
Universidad de las Américas - Puebla, Mexico
72820 Cholula, México,
sobczyk@mail.udlap.mx

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Abstract. This article explores the use of geometric algebra in linear and multilinear algebra, and in affine, projective and conformal geometries. Our principal objective is to show how the rich algebraic tools of geometric algebra are fully compatible with, and augment the more traditional tools of matrix algebra. The novel concept of an h-twistor makes possible a simple new proof of the striking relationship between conformal transformations in a pseudoeuclidean space to isometries in a pseudoeuclidean space of two higher dimensions. The utility of the h-twistor concept, which is a generalization of the idea of a Penrose twistor to a pseudoeuclidean space of arbitrary signature, is amply demonstrated in a new treatment of the Schwarzian derivative.

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Contents

1. Introduction

Almost 125 years after the discovery of "geometric algebra" by William Kingdon Clifford in 1878, the discipline still languishes off the center-stage of mathematics. Whereas Clifford's geometric algebra has gained currency among an increasing number scientists in different "special interest" groups, the authors of the present work contend that geometric algebra should be known by all mathematicians and other scientists for what it really is - the natural algebraic completion of the real number system to include the concept of direction. Whereas, evidently, most mathematicians and other scientists are either unfamiliar with or reject this point of view, we will try to prevail by showing that Clifford algebra really *already has* been universally recognized in the guise of linear algebra. Since linear algebra is *fully compatible* with Clifford algebra, it follows that in learning linear algebra, every scientist has really learned Clifford algebra but is generally unaware of this fact! What is lacking in the standard treatments of linear algebra is the recognition of the natural graded structure of linear algebra and, therefore, the geometric interpretation that goes along with the definition of geometric algebra. As has been often repeated by Hestenes and others, geometric algebra should be seen as a great unifier of the geometric ideas of mathematics (Hestenes, 1991).

The purpose of the present article is to develop the ideas of geometric algebra alongside the more traditional tools of linear algebra by taking full advantage of their fully compatible structures. There are many advantages to such an approach. First, everybody knows matrix algebra, but not everybody is aware that exactly the same algebraic rules apply to the multivectors in a geometric algebra. Because of this fact, it is natural to consider matrices whose elements are taken from a geometric algebra. At the same time, by developing geometric algebra in such a way that any problem can be easily changed into an equivalent problem in matrix algebra, it becomes possible to utilize the powerful and extensive computer software that has been developed for working with matrices. Whereas CLICAL has proven itself to be a powerful computer aid in checking tedious Clifford algebra calculations, it lacks symbolic capabilities (Lounesto, 1994). Geometric algebra offers not only a comprehensive geometric interpretation but also a whole new set of algebraic tools for dealing with problems in linear algebra. We show that matrices, which are rectangular blocks of numbers, represent geometric numbers in a rather special *spinor basis* of a geometric algebra with neutral signature.

This work consists of four main chapters. This introductory chapter lays down the rational for this article and gives a brief summary of its main ideas and content. Chapter 2 is primarily concerned with the development of the basic ideas of linear and multilinear algebra on an n-dimensional real vector space we call the *null space*, since we are assuming that all vectors in $\mathcal{N}$ are *null vectors* (the square of each vector is zero). Taking all linear combinations of sums of products of vectors in $\mathcal{N}$ generates the 2^n-dimensional associative Grassmann algebra $\mathcal{G}(\mathcal{N})$. This stucture is sufficiently rich to efficiently develop many of the basic notions of linear algebra, such as the matrix of a linear operator and the theory of determinants and their properties.

Recently, there has been much interest in the application of geometric algebra to affine, projective and other non-euclidean geometries, (Maks, 1989), (Hestenes, 1991), (Hestenes and Ziegler, 1991), (Porteous, 1995) and (Havel, 1995). These noneuclidean models offer new computational tools for doing pseudeoeucliean and affine geometry using geometric algebra. Chapter 3 undertakes a systematic study of some of these models, and shows how the tools of geometric algebra make it possible to move freely between them, bringing a unification to the subject that is otherwise impossible. One of the key ideas is to define the *meet* and *join* operations on equivalence classes of blades of a geometric algebra which represent subspaces. Since a nonzero r-*blade* characterizes only the *direction* of a subspace, the magnitude of the blade is unimportant. Basic formulas for incidence relationships between points, lines, planes, and higher dimensional objects are compactly formulated. Examples of calculations are given in the affine plane which are just plain fun!

Chapter 4 explores the deep relationships which exist between projective geometry and the conformal group. The conformal geometry of a pseudo-euclidean space can be linearized by considering the horosphere in a pseudo-Euclidean space of two dimensions higher. The introduction of the novel concept of an h-twistor makes possible a simple new proof of the striking relationship between conformal transformations in a pseudoeuclidean space to isometries in a pseudoeuclidean space of two higher dimensions. The concept of an *h-twistor* greatly simplifies calculations and is in many ways a generalization of the successful spinor/twistor formalisms to pseudoeuclidean spaces of arbitrary signatures. The utility of the h-twistor concept is amply demonstrated in a new derivation of the Schwarzian derivative (Davis, 1974, p46), (Nehari, 1952, p199).

2. Geometric Algebra and Matrices

Let $\mathcal{N}$ be an n-dimensional vector space over a given field $\mathcal{K}$, and let

$$\{e\} = (\, e_1 \quad e_2 \quad \cdots \quad e_n \,) \tag{1}$$

be a basis of $\mathcal{N}$. In this work we only consider real ($\mathcal{K} = I\!R$) or complex ($\mathcal{K} = \mathbb{C}$) vector spaces although other fields could be chosen. By interpreting each of the vectors in $\{e\}$ to be the *column vectors* of the *standard basis* of the identity matrix $id(n)$ of the $n \times n$ matrix algebra $\mathcal{M}(\mathcal{K})$ over the field $\mathcal{K}$, we are free to make the identification $\{e\} = id(n)$. We wish to emphasize that we are interpreting the basis vectors e_i to be elements of the $1 \times n$ row matrix (1), and not the elements of a *set*. Thus, in what follows, we are assumming and often will apply the rules of matrix multiplication when dealing with the (generalized) *row vector* of basis vectors $\{e\}$.

Now let $\overline{\mathcal{N}}$ be the *dual* vector space of 1-forms over the the field $\mathcal{K}$, and let $\{\overline{e}\}$ be the *dual basis* of $\overline{\mathcal{N}}$ with respect to the basis $\{e\}$ of $\mathcal{N}$. If we now interpret each of the vectors in $\{\overline{e}\}$ to be the *row vectors* of the standard basis of the identity matrix $id(n)$ of the $n \times n$ matrix algebra $\mathcal{M}(\mathcal{K})$, we can again make the identification $\{\overline{e}\} = id(n)$. Because we wish to be able to interpret the elements of $\{\overline{e}\}$ as row vectors, we will always write the vectors in $\{\overline{e}\}$ in the *column vector form*

$$\{\overline{e}\} = \begin{pmatrix} \overline{e}_1 \\ \overline{e}_2 \\ \cdot \\ \cdot \\ \overline{e}_n \end{pmatrix} \tag{2}$$

We also assume that the column vector $\{\overline{e}\}$ obeys all the rules of matrix addition and multiplication of a $n \times 1$ column vector.

In terms of these bases, any *vector* or *point* $x \in \mathcal{N}$ can be written

$$x = \{e\}x_{\{e\}} = \begin{pmatrix} e_1 & e_2 & \cdot & \cdot & e_n \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ x_n \end{pmatrix} = \sum_{i=1}^{n} x_i e_i \tag{3}$$

for $x_i \in \mathbb{R}$, where

$$x_{\{e\}} = \begin{pmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ x_n \end{pmatrix}$$

are the column vector of *components* of the vector x with respect to the basis $\{e\}$.

Since vectors in $\mathcal{N}$ are represented by column vectors, and vectors $\overline{y} \in \overline{\mathcal{N}}$ by row vectors, we define the operation of *transpose* of the vector x by

$$x^t = (\{e\}x_{\{e\}})^t = x^t_{\{e\}}\{\overline{e}\} = \begin{pmatrix} x_1 & x_2 & \ldots & x_n \end{pmatrix} \begin{pmatrix} \overline{e}_1 \\ \overline{e}_2 \\ \cdot \\ \cdot \\ \overline{e}_n \end{pmatrix} \tag{4}$$

In the case of the complex field $\mathcal{K} = \mathcal{C}$, we have

$$x^*_{\{e\}} = \overline{\begin{pmatrix} x_1 & x_2 & \cdots & x_n \end{pmatrix}} = \begin{pmatrix} \overline{x}_1 \\ \overline{x}_2 \\ \cdot \\ \cdot \\ \overline{x}_n \end{pmatrix}^t \tag{5}$$

The transpose and Hermitian transpose operations allows us to move between the reciprocal vector spaces $\mathcal{N}$ and $\overline{\mathcal{N}}$. Clearly the operation of Hermitian transpose reduces to the ordinary transpose for real vectors.

We now wish to weld together the structures of the matrix algebra $\mathcal{M}(\mathcal{K})$ and the geometric algebras generated by the vectors in the dual null spaces $\mathcal{N}$ and $\overline{\mathcal{N}}$. Following (Doran, Hestenes, Sommen, Van Acker, 1993), we first consider the Grassmann algebra $\mathcal{G}(\mathcal{N})$, generated by taking all linear combinations of sums and products of the elements in the vector space $\mathcal{N} = span\{e\}$ subject to the condition that for each $x \in \mathcal{N}$, $x^2 = xx = 0$. It follows that

$$(x + y)^2 = x^2 + xy + yx + y^2 = xy + yx = 0 \tag{6}$$

or $xy = -yx$ for all x, y in the *null space* $\mathcal{N}$. The geometric algebra $\mathcal{G}(\mathcal{N})$ generated by a null space $\mathcal{N}$ is called the *Grassmann* or *exterior algebra* for the null space $\mathcal{N}$.

As follows from (6), the Grassmann exterior product $a_1 a_2 \ldots a_k$ of k vectors in $\mathcal{N}$ is *antisymmetric* over the interchange of any two of its vectors;

$$a_1 \ldots a_i \ldots a_j \ldots a_k = -a_1 \ldots a_j \ldots a_i \ldots a_k$$

so that the exterior product of *null vectors* is equivalent to the *outer product* of those vectors:

$$a_1 a_2 \ldots a_k = a_1 \wedge a_2 \wedge \ldots \wedge a_k.$$

The 2^n-dimensional *standard basis* $SB\{e\}$ of $\mathcal{G}(\mathcal{N})$, is generated by taking all products of the vectors in the standard basis $\{e\}$ to get $SB\{e\} =$

$$\{1;\ e_1, \ldots, e_n;\ e_{12}, \ldots, e_{(n-1)n}; \ldots;\ e_{1\cdots k}, \ldots, e_{(n-k+1)\cdots n}; \ldots;\ e_{12\cdots n}\} =$$

$$\{\{e_{\overline{0}}\}, \{e_{\overline{1}}\}, \{e_{\overline{2}}\}, \ldots, \{e_{\overline{n}}\}\}\ , \tag{7}$$

where $\{e_{\overline{k}}\} := (\, e_{1\cdots k} \quad \cdots \quad e_{(n-k+1)\cdots n}\,)$ is the $\binom{n}{k}$-dimensional standard basis of k-vectors

$$e_{j_1 j_2 \cdots j_k} \equiv e_{j_1} e_{j_2} \cdots e_{j_k}$$

for the $\binom{n}{k}$ sets of indices $1 \le j_1 < j_2 < \cdots < j_k \le n$. In particular, it is assumed $\{e_{\overline{0}}\} = (1)$ and $\{e_{\overline{1}}\} = \{e\}$. The unique component of $\{e_{\overline{n}}\}$ is the *pseudoscalar* or *volume element* $I := e_{12\cdots n}$. With respect to the standard basis $SB(e)$ any multivector $X \in \mathcal{G}(\mathcal{N})$ can be expressed in the matrix form

$$X = SB\{e\} X_{\{SB\}} \tag{8}$$

where $X_{\{SB\}}$ is the column vector of components

$$X_{\{SB\}} = \begin{pmatrix} x_{\{\overline{0}\}} \\ x_{\{e\}} \\ x_{\{e_{\overline{2}}\}} \\ \cdot \\ \cdot \\ x_{\{e_{\overline{n}}\}} \end{pmatrix}$$

Just as we used the tranpose operation (4) to move from the the null space $\mathcal{N} = span\{e\}$ to the dual null space $\overline{\mathcal{N}}$, we can extend the definition of the transpose to enable us to move from the Grassmann algebra $\mathcal{G}(\mathcal{N})$, to the Grassmann algebra $\mathcal{G}(\overline{\mathcal{N}})$ of the *reciprocal null* *space* $\overline{\mathcal{N}}$. Since multivectors in $\mathcal{G}(\mathcal{N})$ are represented by column vectors, and multivectors $\overline{Y} \in \mathcal{G}(\overline{\mathcal{N}})$ by row vectors, we define the *transpose* $X^t \in \mathcal{G}(\overline{\mathcal{N}})$ by

$$X^t = (SB\{e\}X_{\{SB\}})^t = X^t_{\{SB\}} SB\{\overline{e}\} \tag{9}$$

where $X^t_{\{SB\}}$ is the row vector of components

$$X^t_{\{SB\}} = \begin{pmatrix} x^t_{\{\overline{0}\}} & x^t_{\{e\}} & x^t_{\{e_{\overline{2}}\}} & \cdot & \cdot & x^t_{\{e_{\overline{n}}\}} \end{pmatrix}.$$

The Hermitian transpose is similarly defined when we are dealing with complex multivectors.

The dual basis of multivectors $SB\{\overline{e}\}$ for $\mathcal{G}(\overline{\mathcal{N}})$ are arranged in a column and are defined by

$$SB\{\overline{e}\} = (SB\{e\})^t = \begin{pmatrix} \{1\} \\ \{\overline{e}\} \\ \{\overline{e}_{\overline{2}}\} \\ \cdot \\ \cdot \\ \{\overline{e}_{\overline{n}}\} \end{pmatrix}$$

where

$$\{\overline{e}_{\overline{k}}\} := \begin{pmatrix} \overline{e}_{k\ldots 1} \\ \cdot \\ \cdot \\ \overline{e}_{n\ldots n-k+1} \end{pmatrix} \tag{10}$$

is the $\binom{n}{k}$-dimensional basis of *dual k-vectors* defined by

$$\overline{e}_{j_1 j_2 \ldots j_k} \equiv \overline{e}_{j_1}\overline{e}_{j_2}\cdots\overline{e}_{j_k}$$

for the $\binom{n}{k}$ sets of indices $n \geq j_1 > j_2 > \ldots > j_k \geq 1$.

The dual space $\overline{\mathcal{N}}$ of the space $\mathcal{N}$, and more generally the dual Grassmann algebra $\mathcal{G}(\overline{\mathcal{N}})$ of the Grassmann algebra $\mathcal{G}(\mathcal{N})$, are defined to satisfy the usual properties of the mathematical *dual space*. What (Doran, Hestenes, Sommen, Van Acker, 1993) observed was that these same properties can be faithfully expressed in a larger *neutral geometric algebra* $\mathcal{G}_{n,n}$ (a fomal definition is given below) containing both of these Grassmann algebras as subalgebras, by replacing the *duality* conditions with corresponding *reciprocal* conditions. We accomplish all this by assuming the additional properties

$$e_i^2 = 0 = \overline{e}_i^2,\; e_i e_j = -e_j e_i,\; \overline{e}_i e_j = -e_j \overline{e}_i \text{ (for } i \neq j),\; \text{and } \overline{e}_i \overline{e}_j = -\overline{e}_j \overline{e}_i, \tag{11}$$

together with the *reciprocal relations*

$$\overline{e}_i \cdot e_j = \delta_{i,j} = e_j \cdot \overline{e}_i \tag{12}$$

for all $i, j = 1, 2, \ldots, n$. With this definition, the Grassmann algebra $\mathcal{G}(\overline{\mathcal{N}})$ of the dual space $\overline{\mathcal{N}}$ becomes the natural reciprocal of the Grassmann algebra $\mathcal{G}(\mathcal{N})$. These relations imply that the reciprocal k-vectors and k-forms of Grassmann algebras $\mathcal{G}(\mathcal{N})$ and $\mathcal{G}(\overline{\mathcal{N}})$ satisfy the reciprocal relations

$$\{e_{\overline{k}}\} \cdot \{\overline{e}_{\overline{k}}\} = id(\tbinom{n}{k} \times \tbinom{n}{k}) \ .$$

The *neutral pseudoeuclidean space* $I\!R^{n,n}$ is defined as the linear space which contains *both* the null spaces $\mathcal{N}$ and $\overline{\mathcal{N}}$. Thus,

$$I\!R^{n,n} = \mathcal{N} \oplus \overline{\mathcal{N}} = \{x + \overline{y} | \ x \in \mathcal{N}, \overline{y} \in \overline{\mathcal{N}}\}.$$

Likewise, the 2^{2n}–dimensional associative geometric algebra $\mathcal{G}_{n,n}$ is defined to be the geometric algebra that contains *both* the Grassmann algebras $\mathcal{G}(\mathcal{N})$ and $\mathcal{G}(\overline{\mathcal{N}})$. We write

$$\mathcal{G}_{n,n} = \mathcal{G}(\mathcal{N}) \otimes \mathcal{G}(\overline{\mathcal{N}}) = gen\{e_1, ..., e_n, \overline{e}_1, ..., \overline{e}_n\}, \tag{13}$$

subjected to the relationships (11) and (12).

A simple example will serve to show the interplay between the well-known matrix multiplication and the geometric product in the super matrix algebra $\mathcal{M}(\mathcal{G}_{n,n})$. Recalling the basic geometric product of two vectors x, y,

$$xy = x \cdot y + x \wedge y, \tag{14}$$

we apply the same product to the of row and column basis vectors $\{\overline{e}\}$ and $\{e\}$, and simultaneously employ matrix multiplication, to get the expressions

$$\{\overline{e}\}\{e\} = \{\overline{e}\}\cdot\{e\} + \{\overline{e}\}\wedge\{e\} = id(n\times n) + \begin{pmatrix} \overline{e}_1\wedge e_1 & \overline{e}_1\wedge e_2 & \ldots & \overline{e}_1\wedge e_n \\ \overline{e}_2\wedge e_1 & \overline{e}_2\wedge e_2 & \ldots & \overline{e}_2\wedge e_n \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ \overline{e}_n\wedge e_1 & \overline{e}_n\wedge e_2 & \ldots & \overline{e}_n\wedge e_n \end{pmatrix}$$

where $id(n \times n)$ is the $n \times n$ identity matrix, computed by taking all inner products $\overline{e}_i \cdot e_j$ between the basis vectors of $\{\overline{e}\}$ and $\{e\}$. Similarly,

$$\{e\}\{\overline{e}\} = \{e\} \cdot \{\overline{e}\} + \{e\}\wedge\{\overline{e}\} = \sum_{i=1}^{n} e_i \cdot \overline{e}_i + \sum_{i=1}^{n} e_i \wedge \overline{e}_i = n + \sum_{i=1}^{n} e_i \wedge \overline{e}_i,$$

giving the useful formulas

$$\{e\}\cdot\{\overline{e}\} = n \qquad \text{and} \qquad \{e\}\wedge\{\overline{e}\} = \sum_{i=1}^{n} e_i\wedge\overline{e}_i \tag{15}$$

Because of the metrical structure induced by the reciprocal relationships (12), we can express the components $x_{\{e\}}$ of the vector $x \in \mathcal{N}$ (3) in the form

$$x_{\{e\}} = \begin{pmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ x_n \end{pmatrix} = \begin{pmatrix} \overline{e}_1\cdot x \\ \overline{e}_2\cdot x \\ \cdot \\ \cdot \\ \overline{e}_n\cdot x \end{pmatrix} = \{\overline{e}\}\cdot x$$

Similarly, the components of the reciprocal vector $x^t \in \overline{\mathcal{N}}$ can be found from

$$x^t_{\{e\}} = (\, x_1 \quad x_2 \quad \ldots \quad x_n \,) = x^t\cdot(\, e_1 \quad e_2 \quad \cdot \quad \cdot \quad e_n \,) \tag{16}$$

We call $\mathcal{G}_{n,n}$ the universal geometric algebra of order 2^{2n}. When n is countably infinite, we call $\mathcal{G} = \mathcal{G}_{\infty,\infty}$ the *universal geometric algebra.* The universal algebra $\mathcal{G}$ contains all of the algebras $\mathcal{G}_{n,n}$ as proper subalgebras. In (Doran, Hestenes, Sommen, Van Acker, 1993), $\mathcal{G}_{n,n}$ is called the *mother algebra.*

2.1. NONDEGENERATE GEOMETRIC ALGEBRAS

The standard bases $\{e\}$ and $\{\overline{e}\}$ of the reciprocal null spaces $\mathcal{N}$ and $\overline{\mathcal{N}}$, taken together, are said to make up a *Witt basis* of null vectors (Ablamowicz and Salingaros, 1985) of the neutral pseudoeuclidean space $I\!R^{n,n}$. From the Witt basis, we can construct the *standard orthonormal basis* of $I\!R^{n,n}$ $\{\sigma,\eta\}$ of $\mathcal{G}_{n,n}$,

$$\sigma_i = e_i + \frac{1}{2}\overline{e}_i \qquad \eta_i = e_i - \frac{1}{2}\overline{e}_i \tag{17}$$

for $i = 1, 2, \ldots, n$. Using the defining relationships (12) of the reciprocal frames $\{e\}$ and $\{\overline{e}\}$, we find that these basis vectors satisfy

$$\sigma_i{}^2 = 1 \qquad \eta_i{}^2 = -1 \qquad \eta_i\sigma_j = -\sigma_j\eta_i \qquad \forall\, i,j = 1,\ldots,n$$

$$\sigma_i\sigma_j = -\sigma_j\sigma_i \qquad \eta_i\eta_j = -\eta_j\eta_i \qquad \forall\, i \neq j$$

The basis $\{\sigma\}$ spans a real Euclidean vector space $I\!R^n$ and generates the geometric subalgebra $\mathcal{G}_{n,0}$, whereas $\{\eta\}$ spans an anti–Euclidean space $I\!R^{0,n}$ and generates the geometric subalgebra $\mathcal{G}_{0,n}$. The standard bases (7) of these geometric algebras naturally take the forms

$$SB\{\sigma\} \quad \text{and} \quad SB\{\eta\},$$

so that a general multivector $X \in \mathcal{G}_{n,0}$ can be written

$$X = SB\{\sigma\}X_{SB}$$

and similarly for an $X \in \mathcal{G}_{n,0}$. We can now express the geometric algebra $\mathcal{G}_{n,n}$ as the product of these geometric subalgebras

$$\mathcal{G}_{n,n} = \mathcal{G}_{n,0} \otimes \mathcal{G}_{0,n} = gen\{\sigma_1, ..., \sigma_n, \eta_1, ..., \eta_n\} \; , \tag{18}$$

again only as linear spaces, but not as algebras.

Notice that when we write down the relationship (17), we have given up the possibility of interpreting the vectors in $\{e\}$ and $\{\overline{e}\}$ as column and row vectors, respectively. When working in the nondegenerate geometric algebras $\mathcal{G}_{n,n}$, $\mathcal{G}_{n,0}$ or $\mathcal{G}_{0,n}$, we use the operation of *reversal*. The reversal of any vector $x \in \mathcal{G}_{n,n}$ is defined by $x^\dagger := x$, and for the k-vector $A_k = a_1 \wedge a_2 \wedge \ldots \wedge a_k$,

$${A_k}^\dagger := a_k \wedge a_{k-1} \wedge \ldots \wedge a_1 = (-1)^{k(k-1)/2} A_k.$$

2.2. Spinor basis

One nice application of the above formalism is that it allows us to simply express a natural isomorphism that exists between the neutral geometric algebra $\mathcal{G}_{n,n}$, and the algebra of all real $2^n \times 2^n$ matrices $\mathcal{M}_{I\!R}(2^n)$. To express this isomorphism, we first define 2^n mutually commuting *idempotents*

$$u_i(\pm) = \frac{1}{2}(1 \pm \sigma_i \eta_i) \tag{19}$$

for $i = 1, 2, \ldots, n$.

We can now define 2^n *mutually annihiliating primitive idempotents* for the algebra $\mathcal{G}_{n,n}$,

$$u_{signs} = \prod_{signs} u_i(signs_i) \tag{20}$$

where *signs* is a particular sequence of n $\pm$ signs, and $signs_i$ is the i^{th} sign in the sequence. For example,

$$u_{+++\ldots+} = \prod_{i=1}^{n} u_i(+) \quad \text{and} \quad u_{---\ldots-} = \prod_{i=1}^{n} u_i(-).$$

The primitive idempotents satisfy the following basic properties

– $\sum_{signs}^{2^n} u_{signs} = 1$

– $\sigma_i u_{+++\dots+} = u_{+\dots+-_i+\dots+}\sigma_i$

– $u_{sign_1} u_{sign_2} = \delta_{sign_1\ sign_2} u_{sign_1}$ where $\delta_{sign_1\ sign_2} = 0$ except when $sign_1 = sign_2$ for which $\delta_{sign_1\ sign_2} = 1$.

The above properties are easily verified.

In contrast to the standard basis $SB\{e\}$ of the neutral geometric algebra $\mathcal{G}_{n,n}$, the *spinor basis* of $\mathcal{G}_{n,n}$ is defined to be the $2^n \times 2^n$ multivectors in the matrix

$$SNB(n,n) = SB\{\sigma\}^t u_{+++\dots+} SB\{\sigma\} \tag{21}$$

The simplest example is the spinor basis for the geometric algebra $\mathcal{G}_{1,1}$. The 2^1 primitive idempotents for this geometric algebra are $u_\pm = \frac{1}{2}(1 \pm \sigma\eta)$. Using (21), the spinor basis $SNB(1,1)$ is found to be

$$SB(\sigma)^t u_+ SB(\sigma) = \begin{pmatrix} 1 \\ \sigma \end{pmatrix} u_+ \begin{pmatrix} 1 & \sigma \end{pmatrix} = \begin{pmatrix} u_+ & \sigma u_- \\ \sigma u_+ & u_- \end{pmatrix}$$

The significance of the position of each multivector in the spinor basis, is that its matrix representation corresponds to a 1 in the same position (with zeros everywhere else).

In terms of the spinor basis, any $2^n \times 2^n$ matrix $\mathcal{A}$ represents the corresponding element $A \in \mathcal{G}_{n,n}$ given by

$$A = SB\{\sigma\} u_{+++\dots+} \mathcal{A}\ SB\{\sigma\}^t.$$

The matrix $\mathcal{A}$ associated with the multivector $A \in \mathcal{G}_{n,n}$ is denoted by $\mathcal{A} = [A]$. This association constitutes an *algebra isomorphism*, since $[A + B] = [A] + [B]$ and $[AB] = [A][B]$. Noting that

$$u_{+\dots+}\, SB\{\sigma\}^t SB\{\sigma\}\, u_{+\dots+} = u_{+\dots+}\, id(2^n \times 2^n),$$

it easily follows that

$$\begin{aligned} AB &= SB\{\sigma\}\, u_{+\dots+}\, [A]\, SB\{\sigma\}^t\ SB\{\sigma\}\, u_{+\dots+}\, [B]\, SB\{\sigma\}^t \\ &= SB\{\sigma\}\, u_{+\dots+}\, [A][B]\, SB\{\sigma\}^t \end{aligned} \tag{22}$$

We will use the spinor basis $SNB(1,1)$ for studying conformal transformation in section 4.

2.3. Symmetric and Hermitian Inner Products

Until now we have only considered real geometric algebras and their corresponding real matrices. Any pseudoscalar of the geometric algebra $\mathcal{G}_{n,n}$ will always have a positive square, and will *anticommute* with the vectors in $\mathbb{R}^{n,n}$. If we insisted on dealing only with *real* geometric algebras, we might consider working in the geometric algebra $\mathcal{G}_{n,n+1}$ where the pseudoscalar element i has the desired property that $i^2 = -1$ and is in the center of the algebra (commutes with all multivectors). A *complex vector* $x + iy$ in $\mathcal{G}_{n,n+1}$ consists of the real vector part x and a pseudovector or $(2n)$-blade part iy.

Instead, we choose to directly *complexify* the geometric algebra $\mathcal{G}_{n,n}$ to get the *complex geometric algebra* $\mathcal{G}_{2n}(\mathbb{C})$ (Sobczyk, 1996). Whereas this algebra is isomorphic to $\mathcal{G}_{n,n+1}$, it is somewhat easier to work with than the former. A *complex* vector $z \in \mathbb{C}^{2n}$ has the form $z = x + iy$ where $x, y \in \mathbb{R}^{2n}$. The imaginary unit i, where $i^2 = -1$, is defined to *commute* with all elements in the geometric algebra $\mathcal{G}_{2n}(\mathbb{C})$.

Consider an orthonormal basis $\{\sigma\} \in \mathbb{C}^{2n}$: $\sigma_i \cdot \sigma_j = \delta_{ij}$. The complexified null space $\mathcal{N}(\mathbb{C})$ and its reciprocal null space $\overline{\mathcal{N}}(\mathbb{C})$ are the subspaces spanned by the complex null vectors

$$e_j = \frac{1}{2}(\sigma_j + i\sigma_{n+j}) \qquad \text{and} \qquad \overline{e}_j = \sigma_j - i\sigma_{n+j}$$

for $j = 1, 2, \ldots, n$. This definition is consistent with (17) if we consider $\eta_j = i\sigma_{n+j}$. Thus a null vector $x \in \mathcal{N}(\mathbb{C})$ has the form $x = \{e\}x_{\{e\}}$ for $x_i \in \mathbb{C}$.

Previously we have defined the transposition (4). This operation can be extended to complex vectors in two different ways. The first way is a *linear* extension. We use the term *transposition* for the linear extension, so that the definition (4) is still valid when $x_i \in \mathbb{C}$. The second extension is *antilinear* and is equivalent to *Hermitian conjugation*: $x^* \equiv x^*_{\{e\}}\{\overline{e}\}$. Both operations, Hermitian conjugation and transposition, take us from the complex null space $\mathcal{N}(\mathbb{C})$ to the dual null space $\overline{\mathcal{N}}(\mathbb{C})$, and if the components of x are all real, both reduce to the real transposition. Applied to the components $x_{\{e\}}$, $x^*_{\{e\}}$ is the usual Hermitian transpose of the column vector $x_{\{e\}}$,

$$x^*_{\{e\}} = \overline{(\, x_1 \quad x_2 \quad \cdots \quad x_n \,)} = \begin{pmatrix} \overline{x}_1 \\ \overline{x}_2 \\ \cdot \\ \cdot \\ \overline{x}_n \end{pmatrix}^t \tag{23}$$

We now define the *symmetric inner product* (x, y), and the *Hermitian inner product* $\langle x, y\rangle$, on $\mathcal{N}(\mathbb{C})$. For all $x, y \in \mathcal{N}(\mathbb{C})$, the two products are defined, respectively, by using transposition and Hermition conjugation:

$$(x, y) := x^t \cdot y = x^t_{\{e\}} y_{\{e\}} \qquad \text{and} \qquad \langle x, y \rangle := x^* \cdot y = x^*_{\{e\}} y_{\{e\}} \quad (24)$$

The Hermitian inner product will be used in the next subsection.

2.4. Linear Transformations

Let $\mathcal{N} \oplus \mathcal{N}'$ and $\overline{\mathcal{N}} \oplus \overline{\mathcal{N}}'$ be $(n+n')$-dimensional reciprocal null spaces in $I\!R^{n+n',n+n'}$ with the dual bases $\{e\} \cup \{e'\}$ and $\{\overline{e}\} \cup \{\overline{e}'\}$. Let $f : \mathcal{N} \rightarrow \mathcal{N}'$ be a linear transformation from the null space $\mathcal{N}$ into the null space $\mathcal{N}'$. In light of the previous section, we can consider the null spaces $\mathcal{N}$ and $\mathcal{N}'$ to be over the real or complex numbers. Let

$$Hom(\mathcal{N}, \mathcal{N}') = \{ f : \mathcal{N} \rightarrow \mathcal{N}' |\ \ f \ \text{ is a linear transformation} \}$$

denote the linear space of all homomorphisms from $\mathcal{N}$ to $\mathcal{N}'$, with the usual operation of addition of transformations. Of course, only when $\mathcal{N} = \mathcal{N}'$ is the operation of multiplication (composition) defined.

Given an operator $f \in Hom(\mathcal{N}, \mathcal{N}')$, $y' = f(x) \equiv fx$, the *matrix* $\mathcal{F}$ of f with respect to the bases $\{e\}$ and $\{e'\}$ is defined by

$$f\{e\} \equiv (\, fe_1 \quad \cdots \quad fe_n \,) = (\, e_1 \quad \cdots \quad e'_{n'} \,)\, \mathcal{F} = \{e'\}\mathcal{F}. \qquad (25)$$

Of course, the matrix $\mathcal{F} = (f_{ij})$ is defined by its $n' \times n$ components $f_{ij} = \overline{e}_i \cdot f(e_j) \in \mathcal{C}$ for $i = 1, 2, \ldots, n'$ and $j = 1, 2, \ldots, n$. It follows that $f(e_j) = \sum_{i=1}^{n'} e'_i f_{ij}$. By dotting both sides of the above equation on the left by $\{\overline{e}'\}$, we find the explicit expression

$$\mathcal{F} = \{\overline{e}'\} \cdot \{e'\}\mathcal{F} = \{\overline{e}'\} \cdot f\{e\}\ .$$

Equation (15) can be used to define the *bivector* F of the linear operator f. It is defined by

$$F = f\{e\} \wedge \{\overline{e}'\}$$

and satisfies the property that $f\ x = F \cdot x$ for all $x \in \mathcal{N}$. The bivector of a linear operator makes possible a new theory of linear operators, and is particularly useful in defining the *general linear group* as a Lie group of bivectors with the commutator product, (Fulton and Harris, 1991), (Eds. Bayro and Sobczyk, 2001, pp. 32).

Given the Hermitian inner product (24), the transpose (or Hermitian transpose (23)) $f^* : \mathcal{N}' \rightarrow \mathcal{N}$ of the mapping $f : \mathcal{N} \rightarrow \mathcal{N}'$ is defined by the requirement that for all $x \in \mathcal{N}$ and $y' \in \mathcal{N}'$,

$$\langle x, f^*(y') \rangle = \langle f(x), y' \rangle \quad \Leftrightarrow \quad \mathcal{F}^* \equiv \{\overline{e}\} \cdot f^*\{e'\} = [f\{e\}]^* \cdot \{e'\}\ .$$

Likewise, we can define the transpose relative to the symmetric inner product

$$(x, f^t(y')) = (f(x), y') \quad \Leftrightarrow \quad \mathcal{F}^t \equiv \{\bar{e}\} \cdot f^t\{e'\} = [f\{e\}]^t \cdot \{e'\} \ .$$

2.5. Outermorphism and generalized traces

A linear transformation f naturally extends multilinearly to act on k-blades,

$$f(x \wedge y \wedge \cdots \wedge z) \equiv f(x) \wedge f(y) \wedge \cdots \wedge f(z) \quad \forall x, y, \ldots, z \in \mathcal{N} \ ,$$

and where $f(1) \equiv 1$. Thus extended, $f : \mathcal{G}(\mathcal{N}) \rightarrow \mathcal{G}(\mathcal{N}')$ is called the *outermorphism* of the linear transformation $f : \mathcal{N} \rightarrow \mathcal{N}'$, since it preserves the structure of the outer product:

$$f(A \wedge B + C \wedge D) = f(A) \wedge f(B) + f(C) \wedge f(D) \quad \forall A, B, C, D \in \mathcal{G}(\mathcal{N}) \ .$$

Geometrically, the outermorphism f maps directed areas into directed areas, and more generally, directed k-vectors into directed k-vectors.

A linear transformation from $\mathcal{N}$ into itself is called an *endomorphism.* Let

$$End(\mathcal{N}) = \{f : \mathcal{N} \rightarrow \mathcal{N} | \ \ f \ \text{ is a linear operator}\}$$

denote the *algebra* of all *endomorphisms* on $\mathcal{N}$. The operations of addition and composition of linear operators is well defined for endomorphisms.

The *determinant* $\det f$ of the endomorphism f is defined to be the eigenvalue of the pseudoscalar element $I = e_{12\cdots n}$:

$$f(I) = \det f \ I \qquad \Leftrightarrow \qquad \det f = f(I) \cdot \overline{I}$$

Thus, $\det f$ is the factor by which volume is scaled by f. The *trace* of f is defined by $\mathrm{tr} f := f\{e\} \cdot \{\bar{e}\}$. Given the outermorphism of f we define the *generalized traces* of f by

$$\mathrm{tr}_i f := f\{e_{\bar{i}}\} \cdot \{\bar{e}_{\bar{i}}\} \ .$$

Particular cases are $\mathrm{tr}_0 f = f(1) \cdot 1 = 1$ and $\mathrm{tr}_1 f = \mathrm{tr} f$. The generalized trace of degree n coincides with the determinant: $\mathrm{tr}_n f = f(I) \cdot \overline{I} = \det f$.

A second basis $\{a\}$ of $\mathcal{N}$ is related to the standard basis $\{e\}$ by the application of some endomorphism a

$$\{a\} = a\{e\} = \{e\}\mathcal{A} = (\, e_1 \quad e_2 \quad \cdots \quad e_n \,)\, \mathcal{A} \tag{26}$$

where $\mathcal{A}$ is called the *matrix of transition* from the basis $\{e\}$ to the basis $\{a\}$. Taking the outer product $\bigwedge_{i=1}^{n}\{a\}$ of the basis vectors $\{a\}$, we get

$$\bigwedge_{i=1}^{n}\{a\} \equiv a_1\wedge a_2\wedge\cdots\wedge a_n = a(e_1\wedge e_2\wedge\cdots\wedge e_n) = \det a\ I\ . \qquad (27)$$

We see from (27) that the determinant of the matrix of transition, $\det\mathcal{A} \equiv \det a$, between two bases cannot be zero.

We can now easily construct a dual or reciprocal basis $\{\overline{a}\}$ for the basis $\{a\}$:

$$\overline{a}_i = (-1)^{i+1}\frac{(a_1\wedge\ldots\wedge i^*\wedge\ldots\wedge a_n)\cdot\overline{I}}{\det a} \qquad (28)$$

where i^* means that a_i is omitted from the product. More compactly, using our matrix notation,

$$\{\overline{a}\} = \frac{a(\{\overline{e}\}\cdot I)\cdot\overline{I}}{a(I)\cdot\overline{I}}\ .$$

Checking, we find that

$$\{\overline{a}\}\cdot\{a\} = \frac{[\,a(\{\overline{e}\}\cdot I\,)\cdot\overline{I}\,]\cdot\{a\}}{a(I)\cdot\overline{I}} = \frac{[\,a(\,I\cdot\{\overline{e}\})\wedge a\{e\}\,]\cdot\overline{I}}{a(I)\cdot\overline{I}}$$

$$= \frac{[\,a(\,(\,I\cdot\{\overline{e}\})\wedge\{e\})\,]\cdot\overline{I}}{a(I)\cdot\overline{I}} = \frac{a(I\ (\{\overline{e}\}\cdot\{e\})\,)\cdot\overline{I}}{a(I)\cdot\overline{I}} = \{\overline{e}\}\cdot\{e\} = id$$

We have actually found the inverse of the transition matrix $\mathcal{A}$, given by $\mathcal{A}^{-1} = \{\overline{a}\}\cdot\{e\}$, (Eds. Bayro and Sobczyk, 2001, p.25).

2.6. CHARACTERISTIC POLYNOMIAL

The *characteristic polynomial* of $f:\mathcal{N}\rightarrow\mathcal{N}$ is defined by

$$\varphi_f(\lambda) = \det(\lambda - f) = (\lambda - f)(I)\cdot\overline{I}.$$

The well-known *Caley-Hamilton theorem*, which says that every linear operator satisfies its *characteristic equation*, is a consequence of the identity

$$f[x\wedge\{e_{\overline{n-1}}\}]\cdot\{\overline{e}_{\overline{n-1}}\} = (x\wedge\{e_{\overline{n-1}}\})\cdot\{\overline{e}_{\overline{n-1}}\}\det f = x\det f \qquad (29)$$

When the left side of this identity is expanded we get

$$f[x\wedge\{e_{\overline{n-1}}\}]\cdot\{\overline{e}_{\overline{n-1}}\} = \sum_{i=1}^{n}(-1)^{i+1}f\{e_{\overline{n-i}}\}\cdot\{\overline{e}_{\overline{n-i}}\}\,f^i(x)$$

$$= \sum_{i=1}^{n} (-1)^{i+1} \operatorname{tr}_{n-i} f \; f^i(x) \ . \tag{30}$$

Expressed in terms of the generalized traces of f, the characteristic polynomial is

$$\varphi_f(\lambda) = (\lambda - f)(e_{12\cdots n}) \cdot \overline{e}_{n\cdots 21} = \sum_{i=0}^{n} (-1)^i f\{e_{\overline{i}}\} \cdot \{\overline{e}_{\overline{i}}\} \lambda^{n-i}.$$

Thus, from (29) and (30), we have $\varphi_f(f) = 0$, i.e. f satisfies its characteristic polynomial.

The above equation (29) can also be used to derive a formula for the inverse of f. We get

$$x = f^{-1}(y) = \frac{(y \wedge f\{e_{\overline{n-1}}\}) \cdot \{\overline{e}_{\overline{n-1}}\}}{\det f}.$$

The *minimal polynomial* $\psi_f(\lambda)$ of f is the polynomial of least degree that has the property that $\psi_f(f) = 0$. Taken over the complex numbers $\mathbb{C}$, we can express φ_f and ψ_f in the *factored form*

$$\varphi_f(\lambda) = \prod_{i=1}^{r} (\lambda - \lambda_i)^{n_i} \quad \text{and} \quad \psi_f(\lambda) = \prod_{i=1}^{r} (\lambda - \lambda_i)^{m_i}$$

where $1 \le m_i \le n_i \le n$ for $i = 1, 2, \ldots, r$, and the roots λ_i are all distinct.

The minimal polynomial uniquely determines, up to an ordering of the idempotents, the following *spectral decomposition theorem* of the linear operator f, (Sobczyk, 2001).

THEOREM 1. *If f has the minimal polynomial $\psi(\lambda)$, then a set of commuting mutually annihilating idempotents and corresponding nilpotents $\{(p_i, q_i) | \ i = 1, \ldots, r\}$ can be found such that*

$$f = \sum_{i=1}^{r} (\lambda_i + q_i) p_i,$$

where $\operatorname{rank}(p_i) = n_i$*, and the index of nilpotency* $\operatorname{index}(q_i) = m_i$*, for $i = 1, 2, \ldots, r$. Furthermore, when $m_i = 1$, $q_i = 0$.*

Clearly, the operator f is diagonalizable if and only if it has the spectral form

$$f = \sum_{i=1}^{r} \lambda_i p_i.$$

The spectral decomposition theorem has many different uses and applies equally well to a linear operator or a geometric number, (Sobczyk, 1993, pp357-364), (Sobczyk, 1997; Sobczyk, 1997a). For example, we can immediately define a *generalized inverse* of the operator f by

$$f^{inv} = \sum_{\lambda_i \neq 0} \frac{1}{\lambda_i}(p_i - \frac{q_i}{\lambda_i} + \cdots + (\frac{-q_i}{\lambda_i})^{m_i - 1})$$

satisfying the conditions $ff^{inv} = f^{inv}f = \sum_{\lambda_i \neq 0} p_i$, (Rao and Mitra, 1971, pp.20).

3. Geometric algebra and Non-Euclidean Geometry

Leonardo da Vinci (1452-1519) was one of the first to consider the problems of projective geometry. However, projective geometry was not formally developed until the work "Traité des propriés projectives des figure" of the French mathematician Poncelet (1788-1867), published in 1822. The extrordinary generality and simplicity of projective geometry led the English mathematician Cayley to exclaim: "Projective Geometry is all of geometry" (Young, 1930).

Let $\mathbb{R}^{n+1}$ be an $(n+1)$-dimensional euclidean space and let $\mathcal{G}_{n+1,0}$ be the corresponding geometric algebra. The *directions* or *rays* of non-zero vectors in $\mathbb{R}^{n+1}$ are identified with the points of the n-dimensional projective plane Π^n, (Hestenes and Ziegler, 1991). More precisely, we write

$$\Pi^n \equiv \mathbb{R}^{n+1}/\mathbb{R}^*$$

where $\mathbb{R}^* = \mathbb{R} - \{0\}$. We thus identify *points*, *lines*, *planes*, and higher dimensional *k-planes* in Π^n with 1, 2, 3, and $(k+1)$-dimensional subspaces $\mathcal{S}^r$ of $\mathbb{R}^{n+1}$, where $k \leq n$. To effectively apply the tools of geometric algebra, we need to introduce the new basic operations of *meet* and *join*, (Eds. Bayro and Sobczyk, 2001, p.27).

3.1. The Meet and Joint Operations

The *meet* and *join* operations of projective geometry are most easily defined in terms of the *intersection* and *direct sum* of the subspaces which name the objects in Π^n. On the other hand, each r-dimensional subspace $\mathcal{A}^r$ can be described by a non-zero r-blade $A_r \in \mathcal{G}(\mathbb{R}^{n+1})$. We say that an r-blade A_r *represents*, or is a *representant* of an r-subspace $\mathcal{A}^r$ of $\mathbb{R}^{n+1}$ if and only if

$$\mathcal{A}^r = \{x \in \mathbb{R}^{n+1} | \ \ x \wedge A_r = 0\}. \tag{31}$$

We denote the equivalence class of all nonzero r-blades $A_r \in \mathcal{G}(\mathbb{R}^{n+1})$ which define the subspace $\mathcal{A}^r$ by

$$\{A_r\}_{ray} := \{tA_r \mid t \in \mathbb{R},\ t \neq 0\}. \tag{32}$$

Evidently, every r-blade in $\{A_r\}_{ray}$ is a representant of the subspace $\mathcal{A}^r$. With these definitions, the problem of finding the meet and join is reduced to a problem in geometric algebra of finding the corresponding *meet* and *join* of the $(r+1)$- and $(s+1)$-blades in the geometric algebra $\mathcal{G}(\mathbb{R}^{n+1})$ which represent these subspaces.

Let A_r, B_s and C_t be non-zero blades representing the three subspaces $\mathcal{A}^r$, $\mathcal{B}^s$ and $\mathcal{C}^t$, respectively. We say that

DEFINITION 1. *The t-blade $C_t = A_r \cap B_s$ is the meet of A_r and B_s if there exists a complementary $(r-t)$-blade A_c and a complementary $(s-t)$-blade B_c with the property that $A_r = A_c \wedge C_t$, $B_s = C_t \wedge B_c$, and $A_c \wedge B_c \neq 0$.*

It is important to note that the t-blade $C_t \in \{C_t\}_{ray}$ is not unique and is defined only up to a non-zero scalar factor, which we choose at our own convenience. The existence of the t-blade C_t (and the corresponding complementary blades A_c and B_c) is an expression of the basic relationships that exists between subspaces.

DEFINITION 1.1. *The $(r+s-t)$-blade $D = A_r \cup B_s$, called the join of A_r and B_s is defined by $D = A_r \cup B_s = A_r \wedge B_c$.*

Alternatively, since the join $A_r \cup B_s$ is defined only up to a non-zero scalar factor, we could equally well define D by $D = A_c \wedge B_s$. We use the symbols $\cap$ *intersection* and $\cup$ *direct sum* from set theory to mark this unusual state of affairs. The problem of "meet" and "join" has thus been solved by finding the direct sum and intersection of linear subspaces and their $(r+s-t)$-blade and t-blade representants.

Note that it is only in the special case when $A_r \cap B_s = 0$ that the join can be considered to *reduce* to the outer product. That is

$$A_r \cap B_s = 0 \quad \Leftrightarrow \quad A_r \cup B_s = A_r \wedge B_s$$

However, after the join $I_{A_r \cup B_s} \equiv A_r \cup B_s$ has been found, it can be used to find the meet $A_r \cap B_s$,

$$A_r \cap B_s = A_r \cdot [B_s \cdot I_{A_r \cup B_s}] = [I_{A_r \cup B_s} \cdot A_r] \cdot B_s \tag{33}$$

While the positive definite metric of $\mathbb{R}^{n+1}$ is irrelevant to the definition of the meet and join of subspaces, the formula (33) holds only in $\mathbb{R}^{n+1}$.

A slightly modified version of this formula will hold in any non-degenerate pseudoeuclidean space $I\!R^{p,q}$, where $p+q=n+1$. In this case, after we have found the join $I_{A_r\cup B_s}$, which is a $(r+k)$-blade, we find a reciprocal $(r+k)$-blade $\overline{I}_{A_r\cup B_s}$ with the property that $\overline{I}_{A_r\cup B_s}\cdot I_{A_r\cup B_s}\neq 0$. The meet $A_r\cap B_s$ may then be defined by

$$A_r\cap B_s = A_r\cdot[B_s\cdot\overline{I}_{A_r\cup B_s}] = [\overline{I}_{A_r\cup B_s}\cdot A_r]\cdot B_s \tag{34}$$

3.2. AFFINE AND PROJECTIVE GEOMETRIES

We have seen in the previous section how the meet and join of the n dimensional projective space Π^n can be defined in an $(n+1)$-dimensional euclidean space $I\!R^{n+1}$. There is a very close connection between affine and projective geometries. A projective space can be considered to be an affine space with idealized points at infinity (Young, 1930). Since all the formulas for meet and join remain valid in the pseudoeuclidean space $I\!R^{p,q}$, subject only to (34), we will define the $n=(p+q)$-dimensional affine plane $\mathcal{A}_e(I\!R^{p,q})$ of the null vector $e=\frac{1}{2}(\sigma+\eta)$ in the larger pseudoeuclidean space $I\!R^{p+1,q+1}=I\!R^{p,q}\oplus I\!R^{1,1}$, where $I\!R^{1,1}=span\{\sigma,\eta\}$ for $\sigma^2=1=-\eta^2$. Whereas, effectively, we are only extending the euclidean space $I\!R^{p,q}$ by the null vector e, it is advantageous to work in the geometric algebra $\mathcal{G}_{p+1,q+1}$ of the *non-degenerate* pseudoeuclidean space $I\!R^{p+1,q+1}$.

The affine plane $\mathcal{A}_e^{p,q}:=\mathcal{A}_e(I\!R^{p,q})$ is defined by

$$\mathcal{A}_e(I\!R^{p,q})=\{x_h=x+e|\ \ x\in I\!R^{p,q}\ \}\subset I\!R^{p+1,q+1}, \tag{35}$$

for the null vector $e\in I\!R^{1,1}$. The affine plane $\mathcal{A}_e(I\!R^{p,q})$ has the nice property that $x_h^2=x^2$ for all $x_h\in\mathcal{A}_e(I\!R^{p,q})$, thus preserving the metric structure of $I\!R^{p,q}$. By employing the *reciprocal* null vector $\overline{e}=\sigma-\eta$ with the property that $e\cdot\overline{e}=1$, we can restate definition (35) of $\mathcal{A}_e(I\!R^{p,q})$ in the form

$$\mathcal{A}_e(I\!R^{p,q})=\{y|\ \ y\in I\!R^{p+1,q+1}\ ,\quad y\cdot\overline{e}=1\quad\text{and}\quad y\cdot e=0\ \}\subset I\!R^{p+1,q+1}$$

This form of the definition is interesting because it brings us closer to the definition of the $n=(p+q)$-dimensional *projective plane* .

We summarize here the important properties of the reciprocal null vectors $e=\frac{1}{2}(\sigma+\eta)$ and $\overline{e}=\sigma-\eta$ that will be needed later, and their relationship to the *hyperbolic unit bivector* $u:=\sigma\eta$.

$$e^2=\overline{e}^2=0,\ \ e\cdot\overline{e}=1,\ \ u=\overline{e}\wedge e=\sigma\wedge\eta,\ \ u^2=1 \tag{36}$$

The projective n-plane Π^n can be defined to be the set of all points of the affine plane $\mathcal{A}_e(I\!R^{p,q})$, taken together with idealized points at infinity. Each point $x_h \in \mathcal{A}_e(\mathbb{R}^{p,q})$ is called a *homogeneous representant* of the corresponding point in Π^n because it satisfies the property that $x_h \cdot \overline{e} = 1$. To bring these different viewpoints closer together, points in the affine plane $\mathcal{A}_e(\mathbb{R}^{p,q})$ will also be represented by *rays* in the space

$$\mathcal{A}_e^{rays}(\mathbb{R}^{p,q}) = \{\{y\}_{ray} | \ y \in \mathbb{R}^{p+1,q+1}, y \cdot e = 0, \quad y \cdot \overline{e} \neq 0 \ \} \subset \mathbb{R}^{p+1,q+1} \tag{37}$$

The set of rays $\mathcal{A}_e^{rays}(\mathbb{R}^{p,q})$ gives another definition of the affine n-plane, because each ray $\{y\}_{ray} \in \mathcal{A}_e^{rays}(\mathbb{R}^{p,q})$ determines the unique homogeneous point

$$y_h = \frac{y}{y \cdot \overline{e}} \in \mathcal{A}_e(\mathbb{R}^{p,q}).$$

Conversely, each point $y \in \mathcal{A}_e(\mathbb{R}^{p,q})$ determines a unique ray $\{y\}_{ray}$ in $\mathcal{A}_e^{rays}(\mathbb{R}^{p,q})$. Thus, the affine plane of homogeneous points $\mathcal{A}_e(\mathbb{R}^{p,q})$ is equivalent to the affine plane of rays $\mathcal{A}_e^{rays}(\mathbb{R}^{p,q})$.

Suppose that we are given that we are given k-points $a_1^h, a_2^h, \ldots, a_k^h \in \mathcal{A}_e(\mathbb{R}^{p,q})$ where each $a_i^h = a_i + e$ for $a_i \in \mathbb{R}^{p,q}$. Taking the outer product or *join* of these points gives the projective $(k-1)$-plane $A^h \in \Pi^n$. Expanding the outer product gives

$$\begin{aligned} A^h = a_1^h \wedge a_2^h \wedge \ldots \wedge a_k^h &= a_1^h \wedge (a_2^h - a_1^h) \wedge a_3^h \wedge \ldots \wedge a_k^h \\ &= a_1^h \wedge (a_2^h - a_1^h) \wedge (a_3^h - a_2^h) \wedge a_4^h \wedge \ldots \wedge a_k^h = \ldots \\ &= a_1^h \wedge (a_2 - a_1) \wedge (a_3 - a_2) \wedge \ldots \wedge (a_k - a_{k-1}), \end{aligned}$$

or

$$\begin{aligned} A^h = a_1^h \wedge a_2^h \wedge \ldots \wedge a_k^h &= a_1 \wedge a_2 \wedge \ldots \wedge a_k + \\ e \wedge (a_2 - a_1) \wedge (a_3 - a_2) \wedge \ldots \wedge (a_k - a_{k-1}). \end{aligned} \tag{38}$$

Whereas (38) represents a $(k-1)$-plane in Π^n, it also belongs to the affine (p,q)-plane $\mathcal{A}_e^{p,q}$, and thus contains important metrical information. Dotting this equation with $\overline{e}$, we find that

$$\overline{e} \cdot A^h = \overline{e} \cdot (a_1^h \wedge a_2^h \wedge \ldots \wedge a_k^h) = (a_2 - a_1) \wedge (a_3 - a_2) \wedge \ldots \wedge (a_k - a_{k-1}).$$

This result motivates the following

DEFINITION 1.1.1. *The directed content of the* $(k-1)$*-simplex* $A^h = a_1^h \wedge a_2^h \wedge \ldots \wedge a_k^h$ *in the affine* (p,q)*-plane is given by*

$$\begin{aligned} \frac{\overline{e} \cdot A^h}{(k-1)!} &= \frac{\overline{e} \cdot (a_1^h \wedge a_2^h \wedge \ldots \wedge a_k^h)}{(k-1)!} \\ &= \frac{(a_2 - a_1) \wedge (a_3 - a_2) \wedge \ldots \wedge (a_k - a_{k-1})}{(k-1)!} \end{aligned}$$

3.3. EXAMPLES

Many incidence relations can be expressed in the affine plane $\mathcal{A}_e(I\!R^{p,q})$ which are also valid in the projective plane Π^n, (Eds. Bayro and Sobczyk, 2001, pp.263). A few examples are provided below.

Given 4 coplanar points $a_h, b_h, c_h, d_h \in \mathcal{A}_e(I\!R^2)$. The join and meet of the lines $a_h \wedge b_h$ and $c_h \wedge d_h$ are given, respectively, by $(a_h \wedge b_h) \cup (c_h \wedge d_h) = a_h \wedge b_h \wedge c_h$, and using (34)

$$(a_h \wedge b_h) \cap (c_h \wedge d_h) = [\overline{I} \cdot (a_h \wedge b_h)] \cdot (c_h \wedge d_h)$$

where $\overline{I} = \sigma_2 \wedge \sigma_1 \wedge \overline{e}$. Carrying out the calculations for the meet and join, we find that

$$(a_h \wedge b_h) \cup (c_h \wedge d_h) = \det\{a_h, b_h, c_h\} I = \det\{a, b\} I \qquad (39)$$

where $I = \sigma_1 \wedge \sigma_2 \wedge e$, and

$$(a_h \wedge b_h) \cap (c_h \wedge d_h) = \det\{c - d, b - c\} a_h + \det\{c - d, c - a\} b_h \qquad (40)$$

Note that the meet (40) is not, in general, a homogeneous point. Normalizing (40), we find the homogeneous point $p_h \in \mathcal{A}_e(I\!R^2)$

$$p_h = \frac{\det\{c - d, b - c\} a_h + \det\{c - d, c - a\} b_h}{\det\{c - d, b - a\}}$$

which is the intersection of the lines $a_h \wedge b_h$ and $c_h \wedge d_h$, see Figure 1. The meet can also be solved for directly in the affine plane by noting that

$$p_h = \alpha_p a_h + (1 - \alpha_p) b_h = \beta_p c_h + (1 - \beta_p) d_h$$

and solving to get $\alpha_p = \det\{b_h, c_h, d_h\} / \det\{b_h - a_h, c_h, d_h\}$.

Given the line $a_h \wedge b_h \in \mathcal{A}_e(I\!R^2)$ and a third point $d_h \in \mathcal{A}_e(I\!R^2)$, as in Figure 1, the point f_h on the line $a_h \wedge b_h$ which is closest to the point d_h is called the *foot* of the point d_h on the line $a_h \wedge b_h$. Since $f_h \wedge a_h \wedge b_h = 0$, it follows that $f_h = \alpha_f a_h + (1 - \alpha_f) b_h$ and $f_h \wedge b_h = \alpha_f a_h \wedge b_h$. We can solve this last equation for α_f by dotting it with $\overline{e}$, and invoking the auxilliary condition that $(b - f) \cdot (d - f) = 0$. We get

$$\alpha_f = \frac{(a - b).(d - b)}{(a - b)^2} \qquad (41)$$

It should be carefully noted that $a_h - b_h = a - b \in I\!R^2$ for any two homogeneous points $a_h, b_h \in \mathcal{A}_e^2$. It follows that the foot f_h on the line $a_h \wedge b_h$ is given by

$$f_h = \frac{(b - d) \cdot (b - a) a_h + (a - d) \cdot (a - b) b_h}{(a - b)^2}. \qquad (42)$$

Saying that $a_h, b_h, c_h \in \mathcal{A}_e^2$ are *non-collinear points* is equivalent to the condition $a_h \wedge b_h \wedge c_h \neq 0$. If d_h is any other point in $\mathcal{A}_e^2$, then $d_h \wedge a_h \wedge b_h \wedge c_h = 0$ so that

$$d_h = \alpha_d a_h + \beta_d b_h + (1 - \alpha_d - \beta_d) c_h.$$

By wedging this last equation by $b_h \wedge c_h$ and $a_h \wedge c_h$, respectively, we can easily solve for α_d and β_d, getting

$$\alpha_d = \frac{\det\{d_h, b_h, c_h\}}{\det\{a_h, b_h, c_h\}} \quad \text{and} \quad \beta_d = \frac{\det\{d_h, c_h, a_h\}}{\det\{a_h, b_h, c_h\}} \tag{43}$$

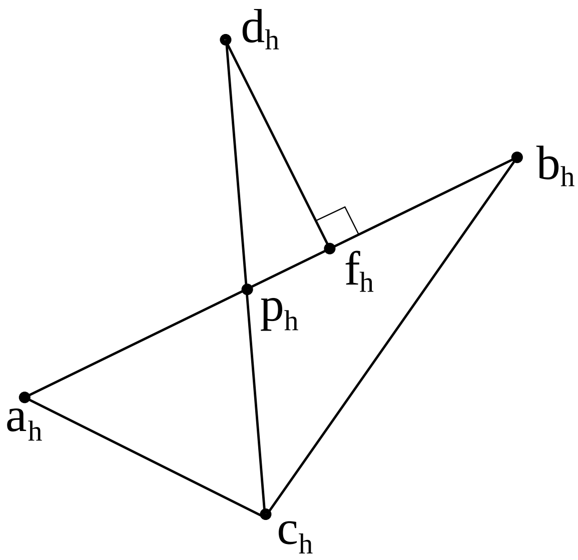

Figure 1. Incidence relationships in the affine plane.

Three non-collinear points $a_h, b_h, c_h \in \mathcal{A}_e^2$ determine a *unique* circle with center $r_h = \alpha_r a_h + \beta_r b_h + (1 - \alpha_r - \beta_r) c_h$. To find the center, note that r_h lies on the intersection of the perperdicular bisectors of the cords $a_h \wedge b_h$ and $a_h \wedge c_h$, and therefore satisfies

$$r_h = \frac{1}{2}(a_h + b_h) + s(c_h - w_h) = \frac{1}{2}(a_h + c_h) + t(b_h - q_h), \tag{44}$$

where

$$w_h = f_w a_h + (1 - f_w) b_h, \quad \text{and} \quad q_h = f_q a_h + (1 - f_q) c_h$$

are the feet (42) of c_h and b_h along the lines $a_h \wedge b_h$ and $a_h \wedge c_h$, respectively, for

$$f_w = \frac{(a - b \cdot (c - b)}{(a - b)^2)} \quad \text{and} \quad f_q = \frac{(c - a) \cdot (b - a)}{(c - a)^2)}.$$

From (44), it follows that

$$(f_w s - f_q t)a_h + [t + (1 - f_w)s - \frac{1}{2}]b_h + [\frac{1}{2} - (1 - f_q)t - s]c_h \equiv 0,$$

which gives

$$s = \frac{f_q}{2f_q(1 - f_w) + 2f_w} \quad \text{and} \quad t = \frac{f_w}{2f_q(1 - f_w) + 2f_w}.$$

After simplification, the center r_h is found to be

$$r_h = \frac{[f_q + f_w - 2f_b f_w]a_h + f_w b_h + f_b c_h}{2[f_q + f_w - f_b f_w]}. \tag{45}$$

Another theorem of interest is Simpson's theorem for the circle. We have assembled all of the tools necessary for a proof of this venerable theorem in the affine plane $\mathcal{A}_e(I\!R^2)$, but we will not prove it here (Eds. Bayro and Sobczyk, 2001, pp.39). Simpson's theorem has also been proven in the non-linear horosphere (Li, Hestenes, and Rockwood, 2000), but the proof is not trivial. It remains to be seen if there are any real advantages to proving such theorems on the horosphere and not in the simpler affine plane. The issue at hand is how to best represent problems in distance geometry (Dress and Havel, 1993).

Hestenes and Zigler have also given a proof of Desargues theorem in the projective plane Π^2 (Hestenes and Ziegler, 1991), by using its representation in the euclidean space $I\!R^3$. A proof of Desargues theorem can also be given in the affine plane of rays $\mathcal{A}_e^{rays}(I\!R^{p,q})$, (Eds. Bayro and Sobczyk, 2001, pp.37). The importance of such proofs is that even though geometric algebra is endowed with a metric, there is no reason why we cannot use the tools of euclidean space to give a proof of this metric independent result. Indeed, as has been emphasized by Hestenes and others (Barnabei, Brini, and Rota, 1985), all the results of linear algebra can be supplied with such a projective interpretation.

4. Conformal Geometry

The conformal geometry of a pseudo-Euclidean space can be linearized by considering the *horosphere* in a pseudo-Euclidean space of two dimensions higher. Because it is so easy to introduce extra orthogonal *anticommuting* vectors into a geometric algebra, without altering the structure of the geometric algebra in any other way, the framework of geometric algebra offers a unification to the subject that is impossible in other formalisms. The horosphere has recently attracted the attention of many workers, see for example, (Dress and Havel, 1993; Porteous, 1995; Havel, 1995).

The horosphere and null cone are formally introduced in subsections 4.1 and 4.2. In subsection 4.3, the concept of an *h-twistor* is introduced which will greatly simplify computations. An h-twistor is a generalization of the Penrose twistor concept. In subsection 4.4, we give a simple proof, using only basic concepts from differential geometry developed in (Hestenes and Sobczyk, 1984), of an intriging result that relates conformal transformations in a pseudoeuclidean space to isometries in a pseudoeuclidean space of two higher dimensions. The original proof of this striking relationship was given by (Haantjes, 1937). In subsection 4.5, we show that for any dimension greater than two, that any isometry on the *null cone* can be extended to all of the pseudoeuclidean space.

In subsections 4.6 and 4.7, we show the beautiful relationships that exists between Mobius transformations (linear fractional transformations) and their 2×2 matrix representation over a suitable geometric algebra. In a final subsection, we explore how all of the formalism developed in the previous sections can be utilized in the characterization of conformal transformations of the pseudoeuclidean space $I\!R^{p,q}$. We develop the theory in a novel way which suggests a non-trivial generalization of the theory of two-component spinors and 4-component twistors. Recall that a conformal transformation preserves angles between tangent vectors at each point (Lounesto and Springer, 1989; Porteous, 1995) . The utility of the h-twistor concept is amply demonstrated in a new derivation of the Schwarzian derivative.

We begin by defining the *horosphere* $\mathcal{H}_e^{p,q}$ in $I\!R^{p+1,q+1}$ by *moving up* from the affine plane $\mathcal{A}_e^{p,q} := \mathcal{A}_e(I\!R^{p,q})$.

4.1. THE HOROSPHERE

Let $\mathcal{G}_{p+1,q+1} = gen(I\!R^{p+1,q+1})$ be the geometric algebra of $I\!R^{p+1,q+1}$, and recall the definition (35) of the affine plane $\mathcal{A}_e^{p,q} := \mathcal{A}_e(I\!R^{p,q}) \subset I\!R^{p+1,q+1}$. Any point $y \in I\!R^{p+1,q+1}$ can be written in the form $y = x + \alpha e + \beta \overline{e}$, where $x \in I\!R^{p,q}$ and $\alpha, \beta \in I\!R$.

The *horosphere* $\mathcal{H}_e^{p,q}$ is most directly defined by

$$\mathcal{H}_e^{p,q} := \{x_c = x_h + \beta \overline{e} \mid x_h \in \mathcal{A}_e^{p,q} \text{ and } x_c^2 = 0.\} \tag{46}$$

With the help of (36), the condition that

$$x_c^2 = (x_h + \beta \overline{e})^2 = x^2 + 2\beta = 0$$

gives us immediately that $\beta := -\frac{x^2}{2}$. Thus each point $x_c \in \mathcal{H}_e^{p,q}$ has the form

$$x_c = x_h - \frac{x_h^2}{2}\overline{e} = x + e - \frac{x^2}{2}\overline{e} = \frac{1}{2}x_h \overline{e} x_h. \tag{47}$$

The last equality on the right follows from

$$\frac{1}{2}x_h\overline{e}x_h = \frac{1}{2}[(x_h\cdot\overline{e})x_h + (x_h\wedge\overline{e})x_h] = x_h - \frac{1}{2}x_h^2\overline{e}.$$

Just as $x_h \in \mathcal{A}_e^{p,q}$ is called the *homogeneous representant* of $x \in I\!R^{p,q}$, the point x_c is called the *conformal representant* of *both* the points $x_h \in \mathcal{A}_e^{p,q}$ and $x \in I\!R^{p,q}$. The set of all conformal representants $\mathcal{H}^{p,q} := c(I\!R^{p,q})$ is called the *horosphere* . The horosphere $\mathcal{H}^{p,q}$ is a non-linear model of both the affine plane $\mathcal{A}_e^{p,q}$ and the pseudoeuclidean space $I\!R^{p,q}$. The horosphere $\mathcal{H}^n$ for the Euclidean space $I\!R^n$ was first introduced by F.A. Wachter, a student of Gauss, (Havel, 1995), and has been recently finding many diverse applications (Eds. Bayro and Sobczyk, 2001, chapter 1, chapter 4, chapter 6).

Defining the bivector $K_x := \overline{e}\wedge x_c = \overline{e}\wedge x_h$, it is easy to get back x_h by the simple projection,

$$x_h = e\cdot K_x \tag{48}$$

and to $x \in I\!R^{p,q}$, by

$$x = u\cdot(u\wedge x_c) = \overline{e}\cdot(e\wedge x_h), \tag{49}$$

using the bivector u defined in (36).

The set of all null vectors $y \in I\!R^{p+1,q+1}$ make up the *null cone*

$$\mathcal{N} := \{y \in I\!R^{p+1,q+1} |\ \ y^2 = 0\}.$$

The subset of $\mathcal{N}$ containing all the representants $y \in \{x_c\}_{ray}$ for any $x \in I\!R^{p,q}$ is defined to be the set

$$\mathcal{N}_0 = \{y \in \mathcal{N} \mid y\cdot\overline{e} \neq 0\} = \cup_{x\in I\!R^{p,q}}\ \{x_c\}_{ray},$$

and is called the *restricted null cone.* The conformal representant of a null ray $\{z\}_{ray}$ is the representant $y \in \{z\}_{ray}$ which satisfies $y\cdot\overline{e} = 1$. The horosphere $\mathcal{H}^{p,q}$ is the parabolic section of the restricted null cone,

$$\mathcal{H}^{p,q} = \{y \in \mathcal{N}_0 \mid y\cdot\overline{e} = 1\},$$

see Figure 2. Thus $\mathcal{H}^{p,q}$ has dimension $n = p + q$.

The null cone $\mathcal{N}$ is determined by the condition $y^2 = 0$, which taking differencials gives

$$y\cdot dy = 0 \quad \Rightarrow \quad x_c\cdot dy = 0\ , \tag{50}$$

where $\{y\}_{ray} = \{x_c\}_{ray}$. Since $\mathcal{N}_0$ is an $(n+1)$-dimensional surface, then (50) is a condition necessary and sufficient for a vector v to belong to the tangent space to the restricted null cone $\mathcal{T}(\mathcal{N}_0)$ at the point y

$$v \in \mathcal{T}(\mathcal{N}_0) \quad \Leftrightarrow \quad x_c\cdot v = 0\ . \tag{51}$$

It follows that the $(n+1)$-pseudoscalar I_y of the tangent space to $\mathcal{N}_0$ at the point y can be defined by $I_y = Ix_c$ where I is the pseudoscalar of $I\!R^{p+1,q+1}$. We have

$$x_c \cdot v = 0 \quad \Leftrightarrow \quad 0 = I(x_c \cdot v) = (Ix_c) \wedge v = I_y \wedge v. \tag{52}$$

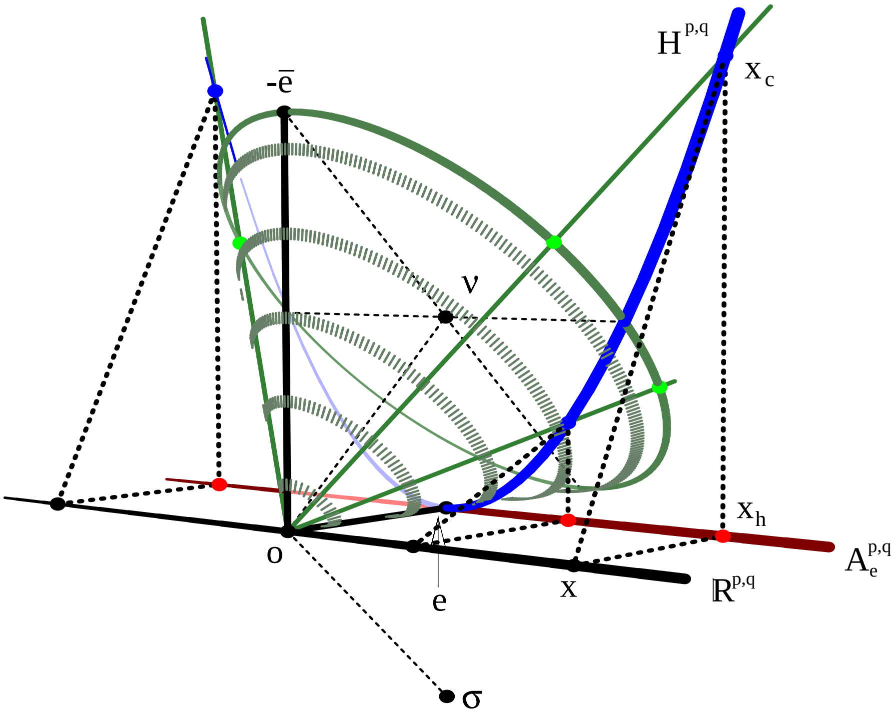

Figure 2. The restricted null cone and representants of the point x in affine space and on the horosphere.

4.2. THE NULL CONE

The mapping

$$c: \ I\!R^{p,q} \hookrightarrow \mathcal{N}_0 \subset I\!R^{p+1,q+1} \ , \quad x \mapsto c(x) \equiv x_c \tag{53}$$

is continuous and infinitely differentiable (indeed, its third differential vanishes), and it is also an isometric embedding.

$$dx_c = dx - x \cdot dx\bar{e} \ \Rightarrow \quad (dx_c)^2 = (dx)^2 \tag{54}$$

The mapping $c(x)$ (53) constitutes a *vectorial chart* for the horosphere. The pseudoscalar I_{x_c} of the *tangent space* to $\mathcal{H}^{p,q}$ at the point x_c is given by

$$I_{x_c} = IK_x = I\ \overline{e}\wedge x_c. \tag{55}$$

We can extend the mapping $c(x)$ to give a *scalar-vector chart* for the whole $\mathcal{N}_0$.

$$y:\ I\!R^{p,q} \times I\!R^* \to \mathcal{N}_0\ ,\quad (x,t) \mapsto y(x,t) \equiv tc(x) = tx_c \tag{56}$$

4.3. H-TWISTORS

Let us define the *h-twistor* to be a rotor $S_x \in \mathrm{Spin}_{p+1,q+1}$

$$S_x := 1 + \frac{1}{2}x\overline{e} = \exp\left(\frac{1}{2}x\overline{e}\right). \tag{57}$$

Noting that $S_x{S_x}^\dagger = 1$, we define its *angular velocity* by

$$\Omega_S := 2{S_x}^\dagger dS_x = dx\overline{e} \qquad \text{or equivalently} \quad \Omega_S(a) = a\overline{e} \quad \forall a \in I\!R^{p,q}\ . \tag{58}$$

Later, in section 4.7, we more carefully define an h-twistor to be an equivalence class of two "twistor" components from $\mathcal{G}_{p,q}$, that have many twistor-like properties.

The reason for these definitions are found in their properties. The point x_c is generated from $0_c = e$ by

$$x_c = S_x e {S_x}^\dagger, \tag{59}$$

and the tangent space to the horosphere at the point x_c is generated from $dx \in I\!R^{p,q}$ by

$$dx_c = dS_x\, e\, {S_x}^\dagger + S_x\, e\, d{S_x}^\dagger = S_x(\Omega_S\cdot e){S_x}^\dagger = S_x dx {S_x}^\dagger \tag{60}$$

or, equivalently, in terms of the argument of the differential

$$dx_c(a) = S_x a {S_x}^\dagger \qquad \forall a \in I\!R^{p,q}.$$

It also keeps unchanged the "point at infinity" $\overline{e}$

$$\overline{e} = S_x \overline{e} {S_x}^\dagger.$$

The motivation for the term "h-twistor" is that it generates both points and tangent vectors on the horosphere from the corresponding objects in $I\!R^{p,q}$. We call the h-twistor (60) "non-rotational" because tangent vectors coincide with the differential of points. More generally, the h-twistor $T_x := S_x R_x$, with $R_x \in \mathrm{Spin}(\mathbb{R}^{p,q})$ generates

$$x_c = T_x e T_x{}^\dagger = S_x e S_x{}^\dagger \quad \text{and} \quad dx_c(R_x a R_x{}^\dagger) = T_x a T_x{}^\dagger.$$

The angular velocity Ω_T of the more general h-twistor T_x is easily calculated

$$\Omega_T := 2T_x{}^\dagger dT_x = R_x{}^\dagger \Omega_S R_x + \Omega_R = R_x{}^\dagger dx R_x \overline{e} + \Omega_R. \qquad (61)$$

The analogy with Penrose twistors is, of course, not complete. We will have more to say about this later.

4.4. Conformal Transformations and Isometries

In this subsection we show that every conformal transformation in $\mathbb{R}^{p,q}$ corresponds to two isometries on the null cone $\mathcal{N}_0$ in $\mathbb{R}^{p+1,q+1}$.

DEFINITION 1. *A conformal transformation in $\mathbb{R}^{p,q}$ is any twicely differentiable mapping between two connected open subsets U and V,*

$$f: \; U \longrightarrow V, \quad x \longmapsto x' = f(x)$$

such that the metric changes by only a conformal factor

$$(df(x))^2 = \lambda(x)(dx)^2, \qquad \lambda(x) \neq 0.$$

If $p \neq q$ then $\lambda(x) > 0$. In the case $p = q$, there exists the posibility that $\lambda(x) < 0$, when the conformal transformations belong to two disjoint subsets. We will only consider the case when $\lambda(x) > 0$.

Recall that $\mathcal{N}_0$ can be coordinized by the vector-scalar chart (56). Using the h-twistor (57), (59) and (60), we obtain the expressions

$$y = S_x\, te\, S_x{}^\dagger \quad \text{and} \quad dy = dtx_c + tdx_c = dtS_x e S_x{}^\dagger + tS_x dx S_x{}^\dagger. \qquad (62)$$

It easily follows that

$$(dy)^2 = t^2(dx_c)^2 = t^2(dx)^2. \qquad (63)$$

DEFINITION 1.1. *An isometry F on $\mathcal{N}_0$ is any twicely differentiable mapping between two connected open subsets U_0 and V_0 in the relative topology of $\mathcal{N}_0$,*

$$F: \; U_0 \longrightarrow V_0, \quad y \longmapsto y' = F(y)$$

which satisfies $(dF(y))^2 = (dy)^2$.

Using the scalar-vector chart $y(x,t) = tx_c$, any mapping in $\mathcal{N}_0$ can be expressed in the form

$$y' = F(y) = t'x'_c = \phi(x,t)f(x,t)_c$$

where $t' = \phi(x,t)$ and $x'_c = f(x,t)_c$ are defined implicitly by F. Using (63), we obtain the result that $y' = F(y)$ is an isometry if and only if

$$(dy')^2 = (dy)^2 \;\Leftrightarrow\; t'^2(dx')^2 = t^2(dx)^2 \;\Leftrightarrow\; (df(x,t))^2 = \frac{t^2}{\phi(x,t)^2}(dx)^2.$$

Since $f(x,t), x \in I\!R^{p,q}$ (non degenerate metric), and the right hand side of this equation does not contain dt, it follows that $f(x,t) = f(x)$ is independent of t. It then follows that $\phi(x) := \phi(x,t)/t$ is also indendent of t. Thus, we can express any isometry $y' = F(y)$ in the form $y' = t\phi(x)f(x)_c$, where $f(x)_c \in \mathcal{N}_0$ is the conformal representant of $f(x) \in I\!R^{p,q}$. This implies that $y' = F(y)$ is an isometry iff

$$y' = t\phi(x)f(x)_c \quad \text{and} \quad (df(x))^2 = (\phi(x))^{-2}(dx)^2.$$

Therefore, $f(x)$ is a conformal transformation with

$$\lambda(x) = \phi(x)^{-2} > 0 \;\leftrightarrow\; \phi(x) = \pm\frac{1}{\sqrt{\lambda(x)}}.$$

4.5. Isometries in $\mathcal{N}_0$

In this section we show that for any dimension greater than 2 any isometry in $\mathcal{N}_0$ is the restriction of an isometry in $I\!R^{p+1,q+1}$. The inverse of the statement is obvious. From the definition of an isometry, $(dF(x))^2 = (dy)^2$. Since $dF(y)$ and dy are vectors in $I\!R^{p+1,q+1}$, $dF(y)$ can be obtained as the result of applying a field of orthogonal transformations to dy,

$$dF(y) = R(y)dyR(y)^{*-1} \tag{64}$$

expressed here through a field of *versors* $R(y) \in Pin_{p+1,q+1} \equiv \{X = a_1a_2\cdots a_n \in \mathcal{G}_{p+1,q+1} \;\mid\; a_i^2 = \pm 1\}$. Note that $R(y)^{*-1} = \pm R(y)^\dagger$, where R^* and $R^\dagger$ denote the *main involution* and the *reversion* respectively. Thus, the result that we must prove is that $R(y)$ is constant, i.e. independent of the point y. This shall guarantee that $F(y)$ is a global rigid isometry.

The fact that the tangent space $\mathcal{T}(\mathcal{N}_0)$ has dimension $n+1$ and a metrically degenerate null direction x_c is sufficient to guarantee that the image of $dF(y)$ defines a unique orthogonal transformation in $\mathbb{R}^{p+1,q+1}$, which determines (up to a sign) the versor $R(y)$.

Previously, we found that any isometry $F(y) = t\phi(x)f(x)_c$ in $\mathcal{N}_0$ is linear in the scalar coordinate t. Taking the exterior derivative, we get

$$dF(y) = \frac{dt}{t}F(y) + td\left(\phi(x)f(x)_c\right) = \frac{dt}{t}F(y) + td\left(\phi(x)S_{f(x)}eS_{f(x)}{}^{\dagger}\right)$$

and using (64) and (62), we also have

$$R(y)dyR(y)^{*-1} \;=\; \tfrac{dt}{t}R(y)yR(y)^{*-1} \;+\; tR(y)S_x dx S_x{}^{\dagger}R(y)^{*-1} \quad .$$

It follows that $R(y)S_xdxS_x{}^{\dagger}R(y)^{*-1} = d\left(\phi(x)S_{f(x)}eS_{f(x)}{}^{\dagger}\right)$ and $F(y) = R(y)yR(y)^{*-1}$, so that $R(y)$ is independent of t. We have now shown that any isometry in $\mathcal{N}_0$ satisfies

$$dF(y) = R(x)dyR(x)^{*-1} \qquad \text{and} \qquad F(y) = R(x)yR(x)^{*-1} \tag{65}$$

where $R(x) \in Pin_{p+1,q+1}$ is solely a function of $x \in \mathbb{R}^{p,q}$. It remains to be shown that $R(x) = R$ is also independent of x so that $F(y) = RyR^{*-1}$ is a global orthogonal transformation in $\mathcal{N}_0 \subset \mathbb{R}^{p+1,q+1}$.

We now slightly generalize the definition of the h-twistor to apply to the rotor $R_x := R(x) \in Pin_{p+1,q+1}$. Letting $T_x := R_xS_x$, we can rewrite (65) in the form

$$dF(y) = T_x(t\,dx + dt\,e)T_x{}^{*-1} \quad \text{and} \quad F(y) = T_x\,te\,T_x{}^{*-1} \tag{66}$$

Analogous to (58) and (61), we define the three bivector valued forms:

$$\Omega_R := 2R_x^{-1}dR_x \ , \quad \Omega_0 := S_x{}^{\dagger}\Omega_R S_x \quad \text{and} \quad \Omega_T := 2T_x^{-1}dT_x \tag{67}$$

From the definition of T_x, we obtain the relation

$$\Omega_T = \Omega_0 + \Omega_S.$$

In order to prove that R_x is constant, let us first impose the integrability condition that the second exterior differential ddF must vanish. Note that in the calculations below we are taking into account both the antisymmetry of exterior forms as well as the non-commutativity of multivectors. Using (65), we find

$$0 = ddF = d(R_x\,dy\,R_x{}^{*-1}) = dR_x\,dy\,R_x{}^{*-1} - R_x\,dy\,dR_x{}^{*-1}$$

$$= \frac{1}{2}R_x\left(\Omega_R\,dy + dy\,\Omega_R\right)R_x{}^{*-1} \qquad \Rightarrow \quad \Omega_R\cdot dy = 0$$

Using (62), this is equivalent to

$$\Omega_R \cdot dy = ({S_x}^\dagger \Omega_R S_x) \cdot ({S_x}^\dagger dy S_x) = \Omega_0 \cdot (t\, dx + dt\, e) = 0 \tag{68}$$

Since $R(x)$ and S_x are independent of t, then Ω_0 does not contain dt. Thus, equation (68) can be separated into two parts,

$$\Omega_0 \cdot (t\, dx + dt\, e) = t\, \Omega_0 \cdot dx + dt\, \Omega_0 \cdot e = 0 \;\Rightarrow \begin{cases} \Omega_0 \cdot dx = 0 \\ \Omega_0 \cdot e = 0 \end{cases} \tag{69}$$

From $\Omega_0 \cdot e = 0$, it follows that the bivector-valued form Ω_0 can be written as

$$\Omega_0(x, a) = v(x, a) \wedge e + B(x, a)$$

where $v(x, a)$ is a vector in $I\!R^{p,q}$, and $B(x, a)$ is a bivector in the geometric algebra $\mathcal{G}^2_{p,q}$ of $I\!R^{p,q}$.

Imposing the first equation in (69) we get

$$\Omega_0(a) \cdot b - \Omega_0(b) \cdot a = 0 \Rightarrow \begin{cases} v(a) \cdot b - v(b) \cdot a = 0 \\ B(a) \cdot b - B(b) \cdot a = 0 \Rightarrow \end{cases}$$

$$\Rightarrow B(a) \cdot (b \wedge c) = B(b) \cdot (a \wedge c) \Rightarrow B(a) = 0 \quad \forall a \in I\!R^{p,q}$$

$$\Rightarrow \Omega_0(a) = v(a) \wedge e. \tag{70}$$

The second integrability condition is found by taking the *exterior derivative* of $\Omega_R = 2R_x^{-1} dR_x$ to find

$$d\Omega_R = 2d{R_x}^{-1} dR_x = 2d{R_x}^{-1} R_x {R_x}^{-1} dR_x = -\frac{1}{2} \Omega_R \Omega_R. \tag{71}$$

But (70) implies $\Omega_R \Omega_R = S_x \Omega_0 \Omega_0 {S_x}^\dagger = 0$, from which it follows that $d\Omega_R = 0$. Next, we write this as an equation in Ω_0, getting:

$$0 = d\Omega_R = d(S_x \Omega_0 {S_x}^\dagger) = S_x \left(d\Omega_0 + \Omega_S \times \Omega_0 \right) {S_x}^\dagger$$

$$\Leftrightarrow \quad d\Omega_0 + \Omega_S \times \Omega_0 = 0.$$

With the help of (70) and (58), we now spit this equation into its three multivector parts:

$$d\Omega_0 + \Omega_S \times \Omega_0 = dv\, e + v \wedge dx + v \cdot dx \;\, e \wedge \overline{e} = 0 \quad \Rightarrow \begin{cases} dv = 0 \\ v \wedge dx = 0 \\ v \cdot dx = 0 \end{cases} \tag{72}$$

The bivector part

$$v \wedge dx = 0 \;\Leftrightarrow\; v(a) \wedge b = v(b) \wedge a$$

differentiates drastically between the dimension $d = 2$, and for the dimensions $d > 2$. When $d > 2$, we can wedge this last expression with the vector $a \in I\!R^{p,q}$ to infer

$$v(a)\wedge b\wedge a = 0 \quad \forall\, b \in I\!R^{p,q} \quad \Rightarrow \quad v(a)\wedge a = 0$$

$$\Rightarrow \quad v(a) = \rho a \quad , \quad \rho \in I\!R,$$

from which follows the desired result

$$\rho a\wedge b = \rho b\wedge a \quad \Rightarrow \quad \rho = 0 \quad \Rightarrow \quad v = 0 \quad \Rightarrow \quad \Omega_0 = 0.$$

Therefore $R(x)$ is constant,

$$\Omega_R = 0 \Rightarrow dR(x) = 0 \Rightarrow R(y) = R = \text{constant}.$$

Thus, $F(y)$ is a global orthogonal transformation in $I\!R^{p+1,q+1}$,

$$F(y) = RyR^{*^{-1}} \quad , \qquad R \in Pin_{p+1,q+1}. \tag{73}$$

Since the group of isometries in $\mathcal{N}_0$ is a double covering of the group of Conformal transformations $Con_{p,q}$ in $I\!R^{p,q}$, and the group $Pin_{p+1,q+1}$ is a double covering of the group of orthogonal transformations $O(p+1, q+1)$, it follows that $Pin_{p+1,q+1}$ is a four-fold covering of $Con_{p,q}$.

The case of $d = 2$ will be treated after introducing the matrix representation of next section.

4.6. Matrix representation

The algebra $\mathcal{G}_{p+1,q+1}$ is isomorphic to $\mathcal{G}_{p,q}\otimes\mathcal{G}_{1,1}$. This isomorphism can be specified by means of the so called *conformal split* (Hestenes, 1991). Evidently, once this isomorphism of algebras is established, we can use the matrix representation introduced in subsection 2.2 for $SNB_{1,1}$, taking into account that the 2×2 matrices are defined over the module $\mathcal{G}_{p,q}$. This identification makes possible a very elegant treatment of the so-called *Vahlen matrices* (Lounesto, 1997; Maks, 1989; Cnops, 1996; Porteous, 1995).

The conformal split does not identify the algebra $\mathcal{G}_{p,q}$ appearing in the isomorphism $\mathcal{G}_{p,q}\otimes\mathcal{G}_{1,1}$ directly with $\mathcal{G}_{p,q} := gen\{I\!R^{p,q}\}$. Instead, the conformal split identifies $\mathcal{G}_{p,q}$ with a subalgebra of $\mathcal{G}_{p+1,q+1}$ generated by a subset of trivectors: $\mathbf{G}_{p,q} := gen\{I\!R^{p,q}\,u\}$, where $u = \sigma\eta$ is the unit bivector orthogonal to $I\!R^{p,q}$, as introduced in (36) and subsection 4.1. This subalgebra has the property that it commutes with $\mathcal{G}_{1,1} = gen\{\sigma, \eta\}$ so that

$$\mathcal{G}_{p+1,q+1} = \mathbf{G}_{p,q} \otimes \mathcal{G}_{1,1}.$$

The multivectors belonging to the subalgebra $\mathbf{G}_{p,q}$ are characterized by

$$\mathbf{A} \in \mathbf{G}_{p,q} \quad \Leftrightarrow \quad \mathbf{A} = A^+ + uA^- \ , \qquad A^+ \in \mathcal{G}^+_{p,q} \text{ and } A^- \in \mathcal{G}^-_{p,q}$$

where A^+ and A^- are, respectively, the even and odd multivectors parts of the multivector $A = A^+ + A^- \in \mathcal{G}_{p,q}$. Thus, we have a direct correspondence between the multivector $\mathbf{A} \in \mathbf{G}_{p,q}$ and the multivector $A \equiv A^+ + A^- \in \mathcal{G}_{p,q}$.

Recall that the idempotents $u_\pm = \frac{1}{2}(1 \pm u)$ of the algebra $\mathcal{G}_{1,1}$, first defined in (20), satisfy the properties given in subsection (2.2):

$$u_+ + u_- = 1 \ , \ \ u_+ - u_- = u \ , \ \ u_+u_- = 0 = u_-u_+ \ , \ \ \sigma u_+ = u_-\sigma \ ,$$

and

$$u = \overline{e}\wedge e \ , \quad u_+ = \frac{1}{2}\overline{e}e \ , \quad u_- = \frac{1}{2}e\overline{e} \ ,$$
$$u\overline{e} = \overline{e} = -\overline{e}u \ , \ \ eu = e = -ue \ , \ \ \sigma u_+ = e \ , \ \ 2\sigma u_- = \overline{e} \ .$$

The representation of $\mathcal{G}_{1,1}$, introduced in the subsection 2.2 using the spinor basis, enables us to write any multivector $G \in \mathcal{G}_{p+1,q+1} = \mathbf{G}_{p,q} \otimes \mathcal{G}_{1,1}$ in the form

$$G = (\,1 \quad \sigma\,)\, u_+ \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix} \begin{pmatrix} 1 \\ \sigma \end{pmatrix}$$

where the entries of the 2×2 matrix are in $\mathbf{G}_{p,q}$. Noting that

$$u_+\mathbf{A} = u_+(A^+ + uA^-) = u_+(A^+ + A^-) = u_+A \ ,$$

makes it possible to work directly with the proper subalgebra $\mathcal{G}_{p,q}$, instead of having to deal with the extra complexity introduced by using the subalgebra $\mathbf{G}_{p,q}$.

It follows that each multivector $G \in \mathcal{G}_{p+1,q+1}$ can be written in the form

$$G = (\,1 \quad \sigma\,)\, u_+[G] \begin{pmatrix} 1 \\ \sigma \end{pmatrix} = Au_+ + Bu_+\sigma + C^*u_-\sigma + D^*u_- \qquad (74)$$

where

$$[G] \equiv \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad \text{for} \ \ A, B, C, D \in \mathcal{G}_{p,q}.$$

The matrix $[G]$ denotes the matrix corresponding to the multivector G, and as a consequence of the general argument given in (22), we have the *algebra isomorphism*

$$[G_1 + G_2] = [G_1] + [G_2] \qquad \text{and} \qquad [G_1G_2] = [G_1][G_2],$$

for all $G_1, G_2 \in \mathcal{G}_{p+1,q+1}$. This result is an example of the unusual fact that a matrix representation is sometimes possible even when the module of components $\mathcal{G}_{p,q}$ does not commute with the subalgebra $\mathcal{G}_{1,1}$. Note, also, the relationships

$$u_+[G] = [u_+G] = \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad [G]u_+ = [Gu_+] = \begin{pmatrix} A & 0 \\ C & 0 \end{pmatrix}.$$

The operation of *reversion* of multivectors translates into the following transpose-like matrix operation:

$$\text{if } \; [G] = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad \text{then} \quad [G]^\dagger := [G^\dagger] = \begin{pmatrix} \overline{D} & \overline{B} \\ \overline{C} & \overline{A} \end{pmatrix}$$

where $\overline{A} = A^{*\dagger}$ is the *Clifford conjugation.*

4.7. h-twistors and Mobius transformations

As seen in section 4.3, the point $x_c \in \mathcal{H}_{p,q}$ can be written in the form (59), $x_c = S_x e {S_x}^\dagger$. More generally, in the subsection 4.5, we saw that any conformal transformation $F(x_c)$ must be of the form

$$sT_x e{T_x}^\dagger = F(x_c) = \phi(x) f(x)_c = \phi(x) S_{f(x)}\, e\, {S_{f(x)}}^\dagger \tag{75}$$

where $s := T_x\overline{T_x} = \pm 1$.

Using the matrix representation of the previous section, for a general multivector $G \in \mathcal{G}_{p+1,q+1}$, we find that

$$\begin{aligned}[GeG^\dagger] &= \begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \overline{D} & \overline{B} \\ \overline{C} & \overline{A} \end{pmatrix} \\ &= \begin{pmatrix} B \\ D \end{pmatrix}\begin{pmatrix} \overline{D} & \overline{B} \end{pmatrix}\end{aligned} \tag{76}$$

where

$$[e] = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad [G] \equiv \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad [G]^\dagger = \begin{pmatrix} \overline{D} & \overline{B} \\ \overline{C} & \overline{A} \end{pmatrix}.$$

The relationship (76) suggests defining the *conformal h-twistor* of the multivector $G \in \mathcal{G}_{p+1,q+1}$ to be

$$[G]_c := \begin{pmatrix} B \\ D \end{pmatrix},$$

which may also be identified with the multivector $G_c := Ge = Bu_+ + D^*e$. The *conjugate* of the conformal h-twistor is then naturally defined by

$$[G]_c^\dagger := \begin{pmatrix} \overline{D} & \overline{B} \end{pmatrix}.$$

conformal h-twistors give us a powerful tool for manipulating the conformal representant and conformal transformations much more efficiently. For example, since x_c is generated by the conformal h-twistor $[S_x]_c$, it follows that

$$[x_c] = [S_x]_c[S_x]_c^\dagger = \begin{pmatrix} x \\ 1 \end{pmatrix} \begin{pmatrix} 1 & -x \end{pmatrix} = \begin{pmatrix} x & -x^2 \\ 1 & -x \end{pmatrix} .$$

Two conformal h-twistors $[G_1]_c$ and $[G_2]_c$ will be said to be *equivalent* if they generate the same multivector, i.e., if

$$[G_1]_c[G_1]_c^\dagger = [G_2]_c[G_2]_c^\dagger .$$

This is equivalent to the condition $G_1\, e\, G_1{}^\dagger = G_2\, e\, G_2{}^\dagger$. Two conformal h-twistors $[G_1]_c$ and $[G_2]_c$ will be said to be *projectively equivalent* if they generate the same direction, i.e., if

$$[G_1]_c[G_1]_c^\dagger = \rho[G_2]_c[G_2]_c^\dagger \quad \text{with} \quad \rho \in I\!R^* .$$

This is equivalent to the condition $\{G_1\, e\, G_1{}^\dagger\}_{ray} = \{G_2\, e\, G_2{}^\dagger\}_{ray}$.

A sufficient condition for two spinor to be projectively equivalent is the following:

$$\text{If } \exists H \in \mathcal{G}_{p,q} \ \text{ such that } \ H\overline{H} \in I\!R \ \text{ and } \ [G_2]_c = [G_1]_c H \tag{77}$$
$$\text{then} \quad [G_2]_c[G_2]_c^\dagger = H\overline{H}\,[G_1]_c[G_1]_c^\dagger.$$

Moreover, it is not difficult to show that if any component A, B, C or D of the two conformal h-twistors $\begin{pmatrix} A \\ B \end{pmatrix}$ and $\begin{pmatrix} C \\ D \end{pmatrix}$ is invertible, then this condition is necessary and sufficient.

We can now write the conformal transformation (75) in its spinorial form

$$[F(x_c)] = \phi(x)[S_{f(x)}]_c[S_{f(x)}]_c^\dagger = s[T_x]_c[T_x]_c^\dagger,$$

from which it follows that $[T_x]_c$ and $[S_{f(x)}]_c$ are projectively equivalent spinors. Since the bottom component of

$$[S_{f(x)}]_c = \begin{pmatrix} f(x) \\ 1 \end{pmatrix}$$

is trivially invertible, the two spinors are equivalent by (77). Letting

$$[T_x]_c = \begin{pmatrix} M \\ N \end{pmatrix},$$

it follows that

$$\begin{pmatrix} M \\ N \end{pmatrix} = \begin{pmatrix} f(x) \\ 1 \end{pmatrix} H \quad \Rightarrow \quad H = N \quad \text{and} \quad f(x) = MN^{-1}, \tag{78}$$

and also that $\phi(x) = sN\overline{N}$.

The beautiful linear fractional expression for the conformal transformation $f(x)$,

$$f(x) = (Ax + B)(Cx + D)^{-1} \tag{79}$$

and

$$\phi(x) = s(Cx + D)(\overline{D} - x\overline{C})$$

is a direct consequence of (78). Since $T_x = RS_x$ for the constant versor (73), $R \in Pin_{p+1,q+1}$, its spinorial form is given by

$$[T_x]_c = [R][S_x]_c = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} x \\ 1 \end{pmatrix} = \begin{pmatrix} Ax + B \\ Cx + D \end{pmatrix} = \begin{pmatrix} M \\ N \end{pmatrix},$$

where

$$[R] = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \ , \quad \text{for constants} \quad A, B, C, D \in \mathcal{G}_{p,q}.$$

The linear fractional expression (79) extends to any dimension and signature the well-known Mobius transformations in the complex plane. The components A, B, C, D of $[R]$ are, of course, subject to the condition that $R \in Pin_{p+1,q+1}$.

Although more difficult to manipulate, our conformal h-twistors are a generalization to any dimension and any signature of the familiar 2-component spinors over the complex numbers, and the 4-component twistors. Penrose's twistor theory (Penrose and MacCallum, 1972) has been discussed in the framework of Clifford algebra by a number of authors, for example see (Ablamowicz and Salingaros, 1985), (Eds. Ablamowicz and Fauser, 2000, pp75-92). In the language of spinors, any null vector $y \in \mathcal{N}$ is the *null pole* of a conformal h-twistor, $[y] = [G]_c[G]_c^\dagger$. Also, two h-twistors will define the same null pole if they differ only by a *phase*, $[G_2]_c = [G_1]_c H$, where $H\overline{H} = 1$. To complete the analogy, note that each conformal h-twistor also defines a *null flag*, i.e. a null bivector, tangent to the null cone $\mathcal{N}$. It easily follows from the expressions (59) and (60) that

$$x_c\, dx_c = S_x\, e\, dx\, S_x{}^\dagger \quad \Rightarrow \quad [x_c\, dx_c] = [S_x]_c\, dx\, [S_x]_c^\dagger \ .$$

Finally, any h-twistor differing only by a rotor $H \in Spin_{p,q}$ will give the same null pole but with a different null flag, the null flag rotated by the rotor H:

$$[S_x]_c\, H\, dx\, \overline{H} [S_x]_c^\dagger$$

4.8. The relative matrix representation

In the two preceding subsections, we have introduced and used a matrix representation of $\mathcal{G}_{p+1,q+1}$, based on the isomorphism $\mathcal{G}_{p+1,q+1} \sim \mathcal{G}_{p,q} \otimes \mathcal{G}_{1,1}$. This matrix representation depends only upon the choice of a fixed spin basis in $\mathcal{G}_{1,1}$, but not on any basis of $\mathcal{G}_{p,q}$. We can introduce an alternative *relative matrix representation* relative to a choosen non null direction $a \in \mathbb{R}^{p,q}$, by a slight modification of the former (74), namely,

$$G = \begin{pmatrix} 1 & \sigma a^{-1} \end{pmatrix} u_+ [G]' \begin{pmatrix} 1 \\ a\sigma \end{pmatrix} , \tag{80}$$

so that

$$[G]' = \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix} [G] \begin{pmatrix} 1 & 0 \\ 0 & a^{-1} \end{pmatrix} .$$

Evidently, this relative representation has the disadvantage of depending on the direction a that is chosen. However, it has the important advantage that the parity of $G \in \mathcal{G}_{p+1,q+1}$ is the same as the parity of the components $A, B, C, D \in \mathcal{G}_{p,q}$, where

$$[G]' = \begin{pmatrix} A & B \\ C & D \end{pmatrix} .$$

Moreover, it is more directly related to complex numbers and to the 4-component twistors of (Penrose and MacCallum, 1972). This relative representation will enable us to relate isometries on N_0 for $d = 2$ with analytic and antianalytic functions over the complex numbers $\mathbb{C}$ or over the *dual numbers* $\mathbb{D}$.

The *vectorial representation* of points is most directly related to the *complex representation* of points via the *paravector representant of x* relative to a, defined by

$$z_x := xa^{-1} \qquad \Leftrightarrow \qquad x = z_x a . \tag{81}$$

Whereas this definition is valid in any dimension, we only consider here the dimension $d = p + q = 2$. The set of relative paravectors, in this case, is the even subalgebra:

$$\{z_x \mid x \in \mathbb{R}^{p,q}\} = \mathcal{G}^0_{p,q} \oplus \mathcal{G}^2_{p,q} = \mathcal{G}^+_{p,q} \qquad \text{for} \quad p + q = 2 .$$

Depending on the signature, the square of the pseudoscalar $I \in \mathcal{G}_{p,q}$ can be either negative ($I^2 = -1$) or positive ($I^2 = 1$). It follows that the algebra $\mathcal{G}^+_{p,q}$ is isomorphic to either the complex numbers $\mathbb{C}$ or to the dual numbers

$$\mathcal{G}^+_{2,0} \simeq \mathcal{G}^+_{0,2} \simeq \mathbb{C} \qquad \text{and} \qquad \mathcal{G}^+_{1,1} \simeq \mathbb{D} .$$

The two vectors $\{a, Ia\} \in \mathbb{R}^{p,q}$ constitutes an orthonormal basis. Relative to this basis, the vector x and its paravector z_x have the coordinate forms

$$x = x^1 a + x^2 I a \qquad \text{and} \qquad z_x = x^1 + x^2 I, \tag{82}$$

where $x^1, x^2 \in \mathbb{R}$.

For example, the relative matrix representation of the conformal representant x_c is

$$[x_c]' = \begin{pmatrix} z_x & -z_x\overline{z_x} \\ 1 & -\overline{z_x} \end{pmatrix} a.$$

The relative matrix representation of the *reversion* of (80) is

$$[G]'^{\dagger} := [G^{\dagger}]' = a^{-1} \begin{pmatrix} \overline{D} & -\overline{B} \\ -\overline{C} & \overline{A} \end{pmatrix} a.$$

Conformal h-twistors can also be defined for the relative matrix representation in the obvious way:

$$[G]_c' := \begin{pmatrix} B \\ D \end{pmatrix} \qquad \text{and} \qquad [G]_c'^{\dagger} := \begin{pmatrix} \overline{D} & -\overline{B} \end{pmatrix},$$

and satisfy

$$[GeG^{\dagger}]' = [G]_c' \, [G]_c'^{\dagger} \, a.$$

4.9. Conformal transformations in dimension 2

Before restricting ourselves in subsection 4.5 to dimensions $d > 2$, we found the expression (70)

$$\Omega_0 = v(a)e,$$

from which we derived the conditions (72). The expression for v can be derived from the versor $T \in Pin_{p+1,q+1}$ which generates (67) $\Omega_T = \Omega_0 + \Omega_S$.

Expressing the bivector (58) $\Omega_S = dx\overline{e}$ in terms of the relative paravectors (81) $z_x = xa^{-1}$, we get

$$\Omega_S = dz_x a\overline{e} \ .$$

From the definition (67) of Ω_T, we find

$$dT = \frac{1}{2}T\Omega_T = \frac{1}{2}T(\Omega_0 + \Omega_S) = \frac{1}{2}T(ve + dz_x \, a\overline{e})$$

where the parity of $T \in Pin_{p+1,q+1}$ is even or odd. Let us define

$$G := \begin{cases} T & \text{, if } T \text{ is even} \\ aT & \text{, if } T \text{ is odd} \end{cases} \tag{83}$$

so that G is always even, and for which it is also true that $dG = \frac{1}{2}G\Omega_T$.

Using the relative matrix representation introduced earlier, we have

$$[G]' \equiv \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad \text{and} \quad [\Omega_T]' = \begin{pmatrix} 0 & 2dz_x \\ -av & 0 \end{pmatrix}$$

where $A, B, C, D \in \mathcal{G}^+_{p,q}$. Note that since the matrix representation (80) is defined in terms of constant vectors, the differential will commute with the representation $[dG]' = d[G]'$. It follows that

$$\begin{pmatrix} dA & dB \\ dC & dD \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} 0 & dz_x \\ -\frac{1}{2}av & 0 \end{pmatrix}.$$

We can split this matrix into two columns, getting

$$\begin{pmatrix} dA \\ dC \end{pmatrix} = -\frac{1}{2}av \begin{pmatrix} B \\ D \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} dB \\ dD \end{pmatrix} = \begin{pmatrix} A \\ C \end{pmatrix} dz_x. \tag{84}$$

Equation (84) implies that, considered as functions over $\mathcal{G}^+_{p,q}$ (isomorphic to $\mathbb{C}$ or to $\mathbb{D}$), the two components B and D are analytic, since their differentials are proportional to dz_x. Therefore, the *derivatives* of these analytic functions are

$$\begin{pmatrix} B' \\ D' \end{pmatrix} = \begin{pmatrix} A \\ C \end{pmatrix} \qquad \text{where} \quad B' := \frac{dB}{dz_x}.$$

This implies, in turn, that the components A and C are also analytic

$$\begin{pmatrix} A \\ C \end{pmatrix} = \begin{pmatrix} B' \\ D' \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} A' \\ C' \end{pmatrix} = -\frac{1}{2}\frac{av}{dz_x}\begin{pmatrix} B \\ D \end{pmatrix}. \tag{85}$$

An immediate consequence of the above equations is that the 1-form av is proportional to dz_x, so that Ω_0 takes the form

$$av = g(z_x)dz_x \qquad \Rightarrow \qquad \Omega_0 = a^{-1}e\, g(z_x)\, dz_x, \tag{86}$$

where $g(z_x)$ is also an analytic function over $\mathcal{G}^+_{p,q}$.

Taking into account the change of representation of the conformal h-twistor $[G]_c$ to $[G]'_c$, the formula (78) for the spinor $[T]'_c = \begin{pmatrix} M \\ N \end{pmatrix}$ becomes $f(x) = MN^{-1}a$. Defining the function

$$\mathrm{f} : \mathcal{G}^+_{p,q} \to \mathcal{G}^+_{p,q}\ , \quad z_x \mapsto \mathrm{f}(z_x) := z_{f(x)} = f(x)a^{-1},$$

we obtain $\mathrm{f}(z_x) = MN^{-1}$.

We must now consider the two cases when T in (83) is either odd or even. If T is even then

$$T = G \quad \Rightarrow \quad \begin{pmatrix} M \\ N \end{pmatrix} = \begin{pmatrix} B \\ D \end{pmatrix} \quad \Rightarrow \quad \mathrm{f}(z_x) = BD^{-1}.$$

If T is odd then

$$T = a^{-1}G \quad \Rightarrow \quad \begin{pmatrix} M \\ N \end{pmatrix} = a^{-1}\begin{pmatrix} B \\ D \end{pmatrix} \quad \Rightarrow \quad \mathrm{f}(z_x) = a^{-1}BD^{-1}a.$$

The function $h(z_x)$ defined by

$$h(z_x) := \begin{cases} \mathrm{f}(z_x) & \text{, if } T \text{ is even} \\ a\,\mathrm{f}(z_x)a^{-1} = \overline{\mathrm{f}(z_x)} & \text{, if } T \text{ is odd} \end{cases}$$

has the property that, regardless of whether T is even or odd,

$$h(z_x) = BD^{-1} \qquad \Rightarrow \qquad B = h(z_x)D\ . \tag{87}$$

Since B and D are analytic, it follows that $h(z_x)$ is also analytic. In the case that T is even, it generates the analytic transformation $\mathrm{f}(z_x) = h(z_x)$ in $\mathcal{G}^+_{p,q}$. On the other hand, in the case that T is odd, it generates the anti-analytic transformation $\mathrm{f}(z_x) = \overline{h(z_x)}$.

Using (87) and (85), we can express $[G]'$ in terms of $h(z_x)$,

$$[G]' = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} (hD)' & hD \\ D' & D \end{pmatrix}\ . \tag{88}$$

The fact that G is in $Spin_{p,q}$ can be used to find an explicit expression for D in terms of $h(z_x)$. Using that $GG^\dagger = \pm 1$,

$$[GG^\dagger]' = \begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} D & -B \\ -C & A \end{pmatrix} = \begin{pmatrix} AD-BC & 0 \\ 0 & AD-BC \end{pmatrix},$$

it follows that $\det[G]' \equiv AD-BC = \pm 1$. From (85) and (87), it directly follows that

$$\pm 1 = AD - BC = 2(B'D - BD') = 2D^2(BD^{-1})' = D^2 h'$$

so that formally we have

$$D = \pm(\pm h')^{-\frac{1}{2}} = \pm\frac{1}{\sqrt{\pm h'}} \tag{89}$$

which, in general, represents *four* solutions.

For the complex case $\mathbb{C} \simeq \mathcal{G}^+_{2,0} \simeq \mathcal{G}^+_{0,2}$, where $I^2 = -1$, the four solutions of (89) are given as usual by

$$D = \frac{k}{\sqrt{h'}}\ , \qquad \text{where} \ \ k = \pm 1, \pm I. \tag{90}$$

The inverse and square roots of the dual number $\pm h' \in \mathbb{D} \simeq \mathcal{G}^+_{1,1}$ in (89), where $I^2 = 1$, are not always well defined. The *inverse* of a dual number $z_x = x_1 + x_2 I \in \mathbb{D}$ is given by

$$z_x^{-1} = \frac{1}{z_x} = \frac{z_x^\dagger}{x_1^2 - x_2^2} = \frac{x_1 - x_2 I}{x_1^2 - x_2^2},$$

so will only exist when $x_1 \neq \pm x_2$. It can be shown that the dual number $\pm h'$ (except in the degenerate case when $h'h'^\dagger = 0$) has exactly one of the four hyperbolic Euler forms (Sobczyk, 1995),

$$\pm h' = \begin{cases} \pm\rho \exp(I\phi) \\ \pm\rho I \exp(I\phi) \end{cases} ,$$

where $\rho = \sqrt{|h'h'^\dagger|}$ and ϕ is the *hyperbolic angle* defined by $\pm h'$. Only in the case when the sign of $\pm h'$ can be chosen such that $\pm h' = \rho \exp(I\phi)$, will $\pm h'$ have four well-defined square roots in $I\!D$. For this case we have

$$D = \frac{k}{\sqrt{\pm h'}} = \frac{k}{\sqrt{\rho}} \exp(-\frac{1}{2}I\phi), \qquad \text{where} \quad k = \pm 1, \pm I. \tag{91}$$

Once we have found D, we also have A, B and C (88)

$$B = \frac{kh}{\sqrt{h'}} , \qquad A = k\frac{(h')^2 - \frac{1}{2}hh''}{(h')^{\frac{3}{2}}} , \qquad C = -k\frac{h''}{2(h')^{\frac{3}{2}}},$$

but it is not, in general, possible to solve for the transformation $h(z_x)$ which corresponds to a given $\Omega_0 = a^{-1}e\, g(z_x)\, dz_x$. However, we can find $g(z_x)$ in terms of the function $h(z_x)$: From (85) and (86), we obtain the second order differential equation for the conformal h-twistor of G,

$$\begin{pmatrix} B'' \\ D'' \end{pmatrix} = -\frac{1}{2} g(z_x) \begin{pmatrix} B \\ D \end{pmatrix} . \tag{92}$$

From (92) and (90) or (91), we have

$$g(z_x) = -2\frac{D''}{D} = \frac{h'''}{h'} - \frac{3}{2}\left(\frac{h''}{h'}\right)^2 .$$

It is recognized that $g(z_x)$ is the Schwarzian derivative of $h(z_x)$, which vanishes whenever $h(z_x)$ is a Möbius transformation. There are many possibilities for the further study of the Schwarzian derivative and its generalizations (Kobayashi and Wada, 2000).

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