QUANTUM HERMITE INTERPOLATION POLYNOMIALS

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Abstract. The concept of Lagrange and Hermite interpolation polynomials can be generalized. The spectral basis of idempotents and nilpotents of a factor ring of polynomials provides a powerful framework for the expression of Lagrange and Hermite interpolation in 1, 2 and higher dimensional spaces. We give a new definition of quantum Lagrange and Hermite interpolation polynomials which works on a countably infinite set of points. Examples are given.

1. Spectral basis of polynomials

Let $\mathbb{R}[x]_h := \mathbb{R}[x]/h(x)$ be the *factor ring* of polynomials generated by the principal ideal of the polynomial $h(x) = \prod_{i=1}^{r}(x-x_i)^{m_i}$, where the $x_i \in \mathbb{R}$ are the distinct real zeros of $h(x)$ with the respective algebraic multiplicities m_i, [3]. As a linear space, the *standard basis* of $\mathbb{R}[x]_h$ can written as a *column*

$$X(x) = \begin{pmatrix} 1 & x & x^2 & \cdots & x^{m-1} \end{pmatrix}^T , \tag{1}$$

of powers of x, where $m := \sum_{i=1}^{r} m_i$. Any polynomial $g(x) \in \mathbb{R}[x]_h$ has the form

$$g(x) = \sum_{i=0}^{m-1} a_i x^i = A_X X(x)$$

where A_X is the *row*

$$A_X := \begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_{m-1} \end{pmatrix}_X$$

of real components of $g(x)$ with respect to the standard basis X.

The *spectral basis* of $\mathbb{R}[x]_h$, the column of m polynomials

$$S(x) := \begin{pmatrix} p_1 & q_1 & \cdots & q_1^{m_1-1} & \ldots & p_r & q_r & \cdots & q_r^{m_r-1} \end{pmatrix}^T , \tag{2}$$

can be defined by

$$S(x) := W^{-1} X(x) , \tag{3}$$

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where W is the *generalized Vandermonde matrix* of the polynomial $h(x)$ [6]. Any element $g(x) \in \mathbb{R}[x]_h$, is easily expressed in the spectral basis $S(x)$,

$$g(x) = A_X X(x) = A_X W W^{-1} X(x) = A_S S(x)$$

where $A_S := A_X W$ is the row of components of $g(x)$ in the spectral basis.

The elements of the spectral basis of the factor ring $\mathbb{R}[x]_h$ enjoy the following simple *rules of multiplication* in $\mathbb{R}[x]_h$:

- $\sum_{i=1}^{r} p_i = 1, \quad p_i p_j = \delta_{ij} p_i, \quad p_i q_i = q_i$

(4)

- $q_i^{m_i - 1} \neq 0, \quad q_i^{m_i} = 0,$

for all $i, j = 1, \ldots, r$. These algebraic properties of the spectral basis are most easily derived with the help of the *euclidean algorithym*, [5].

For example, let $h(x) = (x - x_1)(x - x_2)^3$, and $X = \{1, x, x^2, x^3\}^T$. Then

$$W = \begin{pmatrix} X(x_1) & X(x_2) & X^{(1)}(x_2) & X^{(2)}(x_2) \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & 0 & 0 \\ x_1 & x_2 & 1 & 0 \\ x_1^2 & x_2^2 & 2x_2 & 1 \\ x_1^3 & x_2^3 & 3x_2^2 & 3x_2 \end{pmatrix},$$

where $X^{(k)}(x_i) := \frac{1}{k!} \frac{\partial^k}{\partial x^k} X(x)|_{x=x_i}$ represents the k^{th}-normalized derivative of the column vector $X(x)$ evaluated at $x = x_i$. The spectral basis for $\mathbb{R}[x]_h$ is given by

$$S(x) = W^{-1} X(x) \ .$$

For $x_1 = 2$ and $x_2 = 3$, we find

$$S(x) = \begin{pmatrix} p_1 \\ p_2 \\ q_2 \\ q_2^2 \end{pmatrix} = \begin{pmatrix} 27 - 27x + 9x^2 - x^3 \\ -26 + 27x - 9x^2 + x^3 \\ 24 - 26x + 9x^2 - x^3 \\ -18 + 21x - 8x^2 + x^3 \end{pmatrix}. \tag{5}$$

We also have $X(x) = W S(x)$, or

$$X(x) = \begin{pmatrix} 1 \\ x \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} p_1 + p_2 \\ 2p_1 + 3p_2 + q_2 \\ 2^2 p_1 + 3^2 p_2 + 6q_2 + q_2^2 \\ 2^3 p_1 + 3^3 p_2 + 27 q_2 + 9 q_2^2 \end{pmatrix}.$$

The relationship

$$x = 2p_1 + 3p_2 + q_2 \tag{6}$$

is called the *spectral decomposition* of x in $\mathbb{R}[x]_{(x-2)(x-3)^3}$. Note the spectral decompositions for x^2 and x^3 follow by *squaring* and *cubing* both sides of the expression (6) for the spectral decomposition of x, using the rules of multiplication given above.

2. Hermite Interpolation

Since

$$W^{-1} = \{S(0), S^{(1)}(0), \dots, S^{(m-1)}(0)\} \ ,$$

as follows by successively differentiating (3), and

$$I = \{S(x_1), \dots, S^{(m_1-1)}(x_1); \dots; S(x_r), \dots, S^{(m_r-1)}(x_r)\}$$

is the identity $m \times m$ matrix, it follows that the spectral basis $S(x)$ of the factor ring $\mathbb{R}[x]_h$ is a basis of orthogonal *Hermite polynomials* [1], [6]. If in (3), all the nilpotents $q_1, \dots, q_r$ vanish, then the Hermite polynomials reduce to a basis of Lagrange polynomials, [4].

To find the Hermite interpolation polynomial $g(x) \in \mathbb{R}[x]_{(x-2)(x-3)^3}$ of $f(x) = \frac{4}{4+(2x-5)^2}$ we use (6), getting

$$\begin{aligned} g(x) &:= f(2p_1 + (3+q_2)p_2) = f(2)p_1 + f(3+q_2)p_2 \\ &= f(2)p_1 + [f(3) + f^{(1)}(3)q_2 + f^{(2)}(3)q_2^2]p_2 \\ &= f(2)p_1 + [f(3) + f^{(1)}(3)(x-3) + f^{(2)}(x-3)^2]p_2 \ . \end{aligned}$$

The Hermite interpolation polynomial $g(x)$ of $f(x)$ is a *Taylor series*-like expansion of $f(x)$ around the points $x = 2$ and $x = 3$.

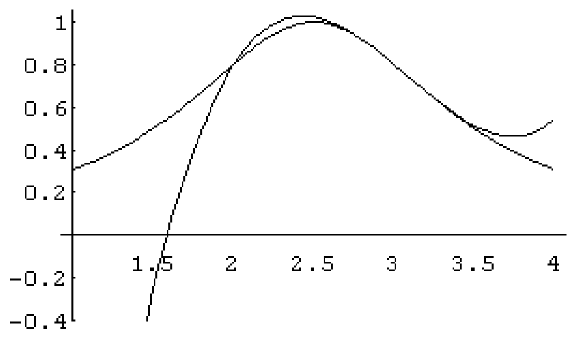

Figure 1: Hermite interpolation $g(x) \in \mathbb{R}[x]_h$ for the function $f(x)$.

3. Hermite Interpolation in Higher Dimension

Let $\{x_1, x_2, \dots, x_r\}$ be interpolation points (knots) along the x-axis with respective multiplicities $\{m_1, \dots, m_r\}$ and let $\{y_1, y_2, \dots, y_s\}$ be interpolation points along the y-axis with respective multiplicities $\{n_1, \dots, n_s\}$. Let

$$h_x = \prod_{i=1}^{r} (x - x_i)^{m_i} \qquad \text{and} \qquad h_y = \prod_{j=1}^{s} (y - y_j)^{n_j}$$

be the corresponding polynomials which define the factor rings $\mathbb{R}[x]_{h_x}$ and $\mathbb{R}[y]_{h_y}$, respectively. The 2-dimensional *Hermite interpolation polynomials* are defined by taking all the products

$$\{p_{x_i} p_{y_j}, p_{x_i} {q_{y_j}}^k, {q_{x_i}}^k p_{y_j}, {q_{x_i}}^k {q_{y_j}}^l\}$$

where $i = 1, \ldots, r$ and $j = 1, \ldots, s$ of the elements of spectral bases of $\mathbb{R}[x]_{h_x}$ and $\mathbb{R}[y]_{h_y}$, [2].

Let $f(x, y)$ be any polynomial defined at the knots (x_i, y_j) to the required orders, $m_i - 1$ and $n_j - 1$ of partial derivatives. The 2-dimensional Hermite interpolation polynomial $g(x, y)$ is defined by

$$\begin{aligned} g(x,y) &= f\Big(\sum_{i=1}^{r}(x_i + q_{x_i})p_i, \sum_{j=1}^{s}(y_j + q_{y_j})p_{y_j}\Big) \\ &= \sum_{i=1}^{r}\sum_{j=1}^{s} f(x_i + q_{x_i}, y_j + q_{y_j}) p_{x_i} p_{y_j} \\ &= \sum_{i=1}^{r}\sum_{j=1}^{s}\Big(\sum_{k=0}^{m_i-1}\sum_{l=0}^{n_j-1}\frac{\partial^{(k)}}{\partial x_i^k}\frac{\partial^{(l)}}{\partial y_i^l} f(x,y)|_{(x_i,y_j)} q_{x_i}^k q_{y_j}^l\Big) p_{x_i} p_{y_j}, \end{aligned}$$

where $\frac{\partial^{(k)}}{\partial x_i^k} := \frac{1}{k!}\frac{\partial^k}{\partial x_i^k}$ and $\frac{\partial^{(l)}}{\partial y_i^l} := \frac{1}{l!}\frac{\partial^l}{\partial y_i^l}$ are the normalized partial derivatives *w.r.t* x_i and y_j, respectively.

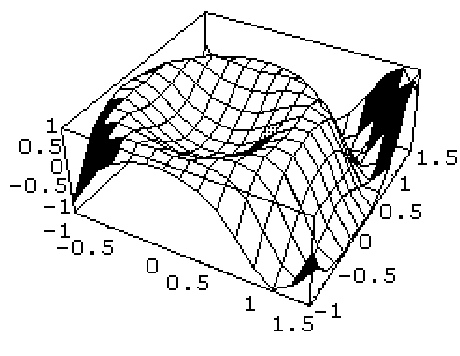

Figure 2: Graph of the function $f(x) = \sin(2x^2 + 2y^2)$.

4. Quantum Hermite interpolation at an infinite number of points

The simplest example of quantum Hermite interpolation is Langrange Interpolation at an infinite number of points. Let $h(x) = \sin(x)$, which has simple zeros at the

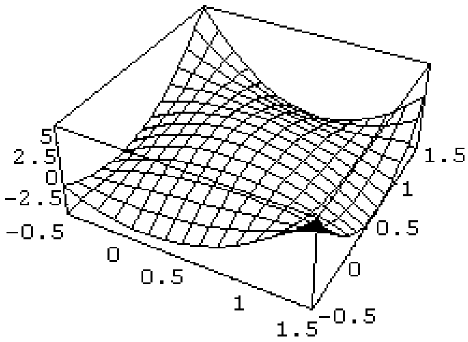

Figure 3: 2-D Hermite interpolation $g(x)$ for the function $f(x)$ for $h_x = x(x-1)^2$ and $h_y = y(y-1)^2$.

points $x_k = k\pi$ for $k = 0, \pm 1, \pm 2, \ldots$. Indeed, it is well-known that the function $\sin x$ can be expressed in the product form

$$\sin x = x \prod_{k=1}^{\infty} (1 - \frac{x^2}{k^2 \pi^2})$$

The *quantum Lagrange interpolation polynomials* are defined by

$$p_k(x) := (-1)^k \frac{\sin(x)}{x - k\pi} \tag{7}$$

for $k = 0, \pm 1, \pm 2, \ldots$. Let $f(x)$ be defined at the knots x_k. The quantum Langrange interpolation polynomial $g(x)$ for the function $f(x)$ is given by

$$g(x) := \sum_{i=-\infty}^{\infty} f(x_i) p_i$$

We use the term "quantum" to reflect the *quantum-like* property that while the quantum interpolation function is defined at only a countable number of points, it is defined non-locally.

More generally, let $K = \{x_\sigma^{m_\sigma} \,|\, \sigma \in Index\}$ be a complete, possibly countably infinite set of the *real* zeros x_σ, with respective (possibly infinite) algebraic multiplicity m_σ, of the real function $h(x)$. The *quantum Hermite interpolation polynomials* for $h(x)$ are defined to be the countably infinite set of polynomials

$$Q_h = \{p_\sigma, \ldots, q_\sigma^{m_\sigma - 1} \,|\, \sigma \in Index\} \tag{8}$$

where p_σ and $q_\sigma := (x - x_\sigma)p_\sigma$ have the same multiplication rules as given for the finite Hermite interpolation polynomials (4).

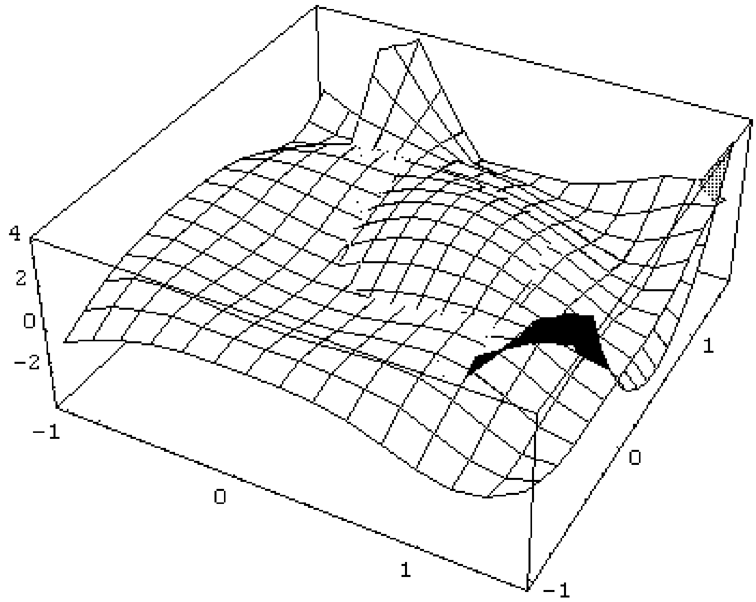

Figure 4: Both the function $f(x)$ and it's Hermite interpolant $g(x)$.

To see these rules in action, and some of the strange quantum-like behavior of these polynomials, let us consider the example $h(x) = \sin^2(1/x)$. A complete set of the real zeros of algebraic multiplicity 2 of the function $h(x)$ is, evidently, $K = \{x_k = \frac{1}{k\pi} \mid k = \pm 1, \pm 2, \ldots\}$. To derive the quantum interpolation polynomials for $h(x)$, we write

$$\sum_{k \in K} p_k = 1 \ . \tag{9}$$

Defining $h_k = h(x)/(x - \frac{1}{k\pi})^2$, and multiplying (9) by h_k, we get $h_k(x)p_k(x) = h_k(x)$. Multiplying each side of this last equation by $h_k^{-1}\ mod(x - \frac{1}{k\pi})^2$, we find that

$$p_k(x) = \frac{h_k(x)}{k^4\pi^4}\Big[1 + 2k\pi(x - \frac{1}{k\pi})\Big] \ .$$

Just as in the case of finite Hermite interpolation polynomials, the spectral decomposition of x is given by

$$x = \sum_{k \in K} x p_k = \sum_{k \in K} (x - x_k + x_k) p_k = \sum_{k \in K} x_k p_k + q_k$$

where

$$q_k := (x - x_k)p_k = (x - x_k)\frac{h_k(x)}{k^4\pi^4}\Big[1 + 2k\pi(x - \frac{1}{k\pi})\Big] = (x - x_k)\frac{h_k(x)}{k^4\pi^4} \ .$$

If $f(x)$ is any function defined at the knots $\{x_k \mid k \in K\}$, then the quantum hermite interpolation function $g(x)$ is defined by

$$g(x) = \sum_{k \in Z^k} f(x_k)p_k + f'(x_k)q_k \ .$$

The figure below shows the graph of both the function $f(x) = 1/x$ and the quantum Hermite interpolation polynomial $g(x)$ for $f(x)$ where $h(x) = \sin^2(1/x)$.

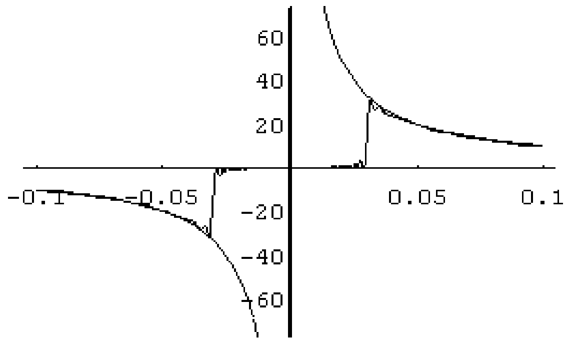

Figure 5: Hermite interpolation $g(x)$ for the function $f(x)$.

References

[1] Bartels, R. H.; Beatty, J. C.; and Barsky, B. A. "Hermite and Cubic Spline Interpolation." Ch. 3 in *An Introduction to Splines for Use in Computer Graphics and Geometric Modelling*, San Francisco, CA: Morgan Kaufmann, pp. 9-17, 1998.

[2] P.J. Davis, *Interpolation and Approximation*, Dover Publications, New York, 1975.

[3] J.A. Gallian, *Contemporary Abstract Algebra*, 4^{th} Edition, Houghton Mifflin Company, Boston, 1998.

[4] Jeffreys, H. and Jeffreys, B. S. "Lagrange's Interpolation Formula." §9.011 in *Methods of Mathematical Physics*, 3rd ed. Cambridge, England: Cambridge University Press, p. 260, 1988.

[5] G. Sobczyk, The Missing Spectral Basis in Algebra and Number Theory, *The American Mathematical Monthly*, Vol. 108, No. 4, April 2001, pp. 336-346.

[6] G. Sobczyk, Generalized Vandermonde Determinants and Applications, *Aportaciones Matemáticas*, Serie Comunicaciones 30 (2002), pp. 41-53.

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