QUANTUM HERMITE INTERPOLATION POLYNOMIALS

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Abstract. The concept of Lagrange and Hermite interpolation polynomials can be generalized. The spectral basis of idempotents and nilpotents of a factor ring of polynomials provides a powerful framework for the expression of Lagrange and Hermite interpolation in 1, 2 and higher dimensional spaces. We give a new definition of quantum Lagrange and Hermite interpolation polynomials which works on a countably infinite set of points. Examples are given.

1. Spectral basis of polynomials

Let $\mathbb{R}[x]_h := \mathbb{R}[x]/h(x)$ be the factor ring of polynomials generated by the principal ideal of the polynomial $h(x) = \prod_{i=1}^{r} (x - x_i)^{m_i}$, where the $x_i \in \mathbb{R}$ are the distinct real zeros of $h(x)$ with the respective algebraic multiplicities $m_i$, [3]. As a linear space, the standard basis of $\mathbb{R}[x]_h$ can written as a column

$$X(x) = \begin{pmatrix} 1 & x & x^2 & \cdots & x^{m-1} \end{pmatrix}^T,$$

of powers of $x$, where $m := \sum_{i=1}^{r} m_i$. Any polynomial $g(x) \in \mathbb{R}[x]_h$ has the form

$$g(x) = \sum_{i=0}^{m-1} a_i x^i = A_X X(x)$$

where $A_X$ is the row

$$A_X := (a_0 \ a_1 \ a_2 \ \cdots \ a_{m-1})_X$$

of real components of $g(x)$ with respect to the standard basis $X$.

The spectral basis of $\mathbb{R}[x]_h$, the column of $m$ polynomials

$$S(x) := (p_1 \ q_1 \ \cdots \ q_1^{m_1-1} \ \cdots \ p_r \ q_r \ \cdots \ q_r^{m_r-1})^T,$$

can be defined by

$$S(x) := W^{-1} X(x),$$

where $W$ is the Vandermonde determinant of $X(x)$.

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where $W$ is the generalized Vandermonde matrix of the polynomial $h(x)$ [6]. Any element $g(x) \in \mathbb{R}[x]_h$, is easily expressed in the spectral basis $S(x)$,

$$g(x) = A_X X(x) = A_X W W^{-1} X(x) = A_S S(x)$$

where $A_S := A_X W$ is the row of components of $g(x)$ in the spectral basis.

The elements of the spectral basis of the factor ring $\mathbb{R}[x]_h$ enjoy the following simple rules of multiplication in $\mathbb{R}[x]_h$:

- $\sum_{i=1}^{r} p_i = 1$, $p_i p_j = \delta_{ij} p_i$, $p_i q_i = q_i$

- $q_i^{n_i - 1} \neq 0$, $q_i^{n_i} = 0$, for all $i, j = 1, \ldots, r$. These algebraic properties of the spectral basis are most easily derived with the help of the euclidean algorithm, [5].

For example, let $h(x) = (x - x_1)(x - x_2)^2$, and $X = \{1, x, x^2, x^3\}^T$. Then

$$W = \begin{pmatrix} X(x_1) & X(x_2) & X^{(1)}(x_2) & X^{(2)}(x_2) \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & 0 & 0 \\
1 & x_2 & 1 & 0 \\
x_1^2 & x_2^2 & 2x_2 & 1 \\
x_1^3 & x_2^3 & 3x_2^2 & 3x_2 \end{pmatrix}.$$

where $X^{(k)}(x_i) := \frac{\partial^k}{\partial x^k} X(x)|_{x=x_i}$ represents the $k$th-normalized derivative of the column vector $X(x)$ evaluated at $x = x_i$. The spectral basis for $\mathbb{R}[x]_h$ is given by

$$S(x) = W^{-1} X(x).$$

For $x_1 = 2$ and $x_2 = 3$, we find

$$S(x) = \begin{pmatrix} p_1 \\
p_2 \\
q_2 \\
q_2^3 \end{pmatrix} = \begin{pmatrix} 27 - 27x + 9x^2 - x^3 \\
-26 + 27x - 9x^2 + x^3 \\
24 - 26x + 9x^2 - x^3 \\
-18 + 21x - 8x^2 + x^3 \end{pmatrix}. \quad (5)$$

We also have $X(x) = W S(x)$, or

$$X(x) = \begin{pmatrix} x \\
x^2 \\
x^3 \end{pmatrix} = \begin{pmatrix} p_1 + p_2 \\
2p_1 + 3p_2 + q_2 \\
2^2p_1 + 3^2p_2 + 6q_2 + q_2^2 \\
2^3p_1 + 3^3p_2 + 27q_2 + 9q_2^2 \end{pmatrix}. \quad (6)$$

The relationship

$$x = 2p_1 + 3p_2 + q_2 \quad (6)$$

is called the spectral decomposition of $x$ in $\mathbb{R}[x]_{(x-2)(x-3)^3}$. Note the spectral decompositions for $x^2$ and $x^3$ follow by squaring and cubing both sides of the expression (6) for the spectral decomposition of $x$, using the rules of multiplication given above.
2. Hermite Interpolation

Since

$$W^{-1} = \{S(0), S^{(1)}(0), \ldots, S^{(m-1)}(0)\}$$

as follows by successively differentiating (3), and

$$I = \{S(x_1), \ldots, S^{(m_1-1)}(x_1); \ldots; S(x_r), \ldots, S^{(m_r-1)}(x_r)\}$$

is the identity $m \times m$ matrix, it follows that the spectral basis $S(x)$ of the factor ring $\mathbb{R}[x]_h$ is a basis of orthogonal Hermite polynomials [1], [6]. If in (3), all the nilpotents $q_1, \ldots, q_r$ vanish, then the Hermite polynomials reduce to a basis of Lagrange polynomials, [4].

To find the Hermite interpolation polynomial $g(x) \in \mathbb{R}[x]_{(x-2)(x-3)^3}$ of $f(x) = \frac{4}{x+2x-3}$ we use (6), getting

$$g(x) := f(2p_1 + (3 + q_2)p_2) = f(2)p_1 + f(3 + q_2)p_2$$

$$= f(2)p_1 + [f(3) + f^{(1)}(3)q_2 + f^{(2)}(3)q_2^2]p_2$$

$$= f(2)p_1 + [f(3) + f^{(1)}(3)(x-3) + f^{(2)}(x-3)^2]p_2.$$ 

The Hermite interpolation polynomial $g(x)$ of $f(x)$ is a Taylor series-like expansion of $f(x)$ around the points $x = 2$ and $x = 3$.

![Figure 1: Hermite interpolation $g(x) \in \mathbb{R}[x]_h$ for the function $f(x)$.

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3. Hermite Interpolation in Higher Dimension

Let $\{x_1, x_2, \ldots, x_r\}$ be interpolation points (knots) along the $x$-axis with respective multiplicities $\{m_1, \ldots, m_r\}$ and let $\{y_1, y_2, \ldots, y_s\}$ be interpolation points along the $y$-axis with respective multiplicities $\{n_1, \ldots, n_s\}$. Let

$$h_x = \prod_{i=1}^{r} (x - x_i)^{m_i} \quad \text{and} \quad h_y = \prod_{j=1}^{s} (y - y_j)^{n_j}$$
be the corresponding polynomials which define the factor rings \( \mathbb{R}[x]_{h_x} \) and \( \mathbb{R}[y]_{h_y} \), respectively. The 2-dimensional Hermite interpolation polynomials are defined by taking all the products
\[
\{ p_{x_i} p_{y_j}, p_{x_i} q_{y_j}, q_{x_i} p_{y_j}, q_{x_i} q_{y_j} \}
\]
where \( i = 1, \ldots, r \) and \( j = 1, \ldots, s \) of the elements of spectral bases of \( \mathbb{R}[x]_{h_x} \) and \( \mathbb{R}[y]_{h_y} \), [2].

Let \( f(x, y) \) be any polynomial defined at the knots \((x_i, y_j)\) to the required orders, \( m_i - 1 \) and \( n_j - 1 \) of partial derivatives. The 2-dimensional Hermite interpolation polynomial \( g(x, y) \) is defined by
\[
g(x, y) = f(\sum_{i=1}^{r} (x_i + q_{x_i})p_{x_i}, \sum_{j=1}^{s} (y_j + q_{y_j})p_{y_j})
\]
\[
= \sum_{i=1}^{r} \sum_{j=1}^{s} f(x_i + q_{x_i}, y_j + q_{y_j})p_{x_i}p_{y_j}
\]
\[
= \sum_{i=1}^{r} \sum_{j=1}^{s} \left( \sum_{k=0}^{m_i-1} \sum_{l=0}^{n_j-1} \partial^{(k)}(x) \partial^{(l)}(y) f(x, y)|_{(x_i, y_j)} q_{x_i}^k q_{y_j}^l \right) p_{x_i}p_{y_j},
\]
where \( \partial^{(k)}(x) := \frac{1}{k!} \frac{\partial^k}{\partial x^k} \) and \( \partial^{(l)}(y) := \frac{1}{l!} \frac{\partial^l}{\partial y^l} \) are the normalized partial derivatives w.r.t \( x_i \) and \( y_j \), respectively.

![Graph of the function \( f(x) = \sin(2x^2 + 2y^2) \).](image)

4. **Quantum Hermite Interpolation at an Infinite Number of Points**

The simplest example of quantum Hermite interpolation is Langrange Interpolation at an infinite number of points. Let \( h(x) = \sin(x) \), which has simple zeros at the
Figure 3: 2-D Hermite interpolation $g(x)$ for the function $f(x)$ for $h_x = x(x-1)^2$ and $h_y = y(y-1)^2$.

points $x_k = k\pi$ for $k = 0, \pm 1, \pm 2, \ldots$. Indeed, it is well-known that the function $\sin x$ can be expressed in the product form

$$\sin x = x \prod_{k=1}^{\infty} \left(1 - \frac{x^2}{k^2 \pi^2}\right)$$

The quantum Lagrange interpolation polynomials are defined by

$$p_k(x) := (-1)^k \frac{\sin(x)}{x - k\pi}$$

for $k = 0, \pm 1, \pm 2, \ldots$. Let $f(x)$ be defined at the knots $x_k$. The quantum Lagrange interpolation polynomial $g(x)$ for the function $f(x)$ is given by

$$g(x) := \sum_{i=-\infty}^{\infty} f(x_i) p_i$$

We use the term “quantum” to reflect the quantum-like property that while the quantum interpolation function is defined at only a countable number of points, it is defined non-locally.

More generally, let $K = \{x_{\sigma}^{m_{\sigma}} \mid \sigma \in \text{Index}\}$ be a complete, possibly countably infinite set of the real zeros $x_{\sigma}$, with respective (possibly infinite) algebraic multiplicity $m_{\sigma}$, of the real function $h(x)$. The quantum Hermite interpolation polynomials for $h(x)$ are defined to be the countably infinite set of polynomials

$$Q_h = \{p_{\sigma}, \ldots, q_{\sigma}^{m_{\sigma}-1} \mid \sigma \in \text{Index}\}$$

where $p_{\sigma}$ and $q_{\sigma} := (x - x_{\sigma})p_{\sigma}$ have the same multiplication rules as given for the finite Hermite interpolation polynomials (4).
Figure 4: Both the function $f(x)$ and it’s Hermite interpolant $g(x)$.

To see these rules in action, and some of the strange quantum-like behavior of these polynomials, let us consider the example $h(x) = \sin^2(1/x)$. A complete set of the real zeros of algebraic multiplicity 2 of the function $h(x)$ is, evidently, $K = \{x_k = \frac{1}{k\pi} | k = \pm 1, \pm 2, \ldots\}$. To derive the quantum interpolation polynomials for $h(x)$, we write

$$\sum_{k \in K} p_k = 1. \quad (9)$$

Defining $h_k = h(x)/(x - \frac{1}{k\pi})^2$, and multiplying (9) by $h_k$, we get $h_k(x)p_k(x) = h_k(x)$. Multiplying each side of this last equation by $h_k^{-1} \mod (x - \frac{1}{k\pi})^2$, we find that

$$p_k(x) = \frac{h_k(x)}{k^4\pi^4} \left[1 + 2k\pi(x - \frac{1}{k\pi})\right].$$

Just as in the case of finite Hermite interpolation polynomials, the spectral decomposition of $x$ is given by

$$x = \sum_{k \in K} xp_k = \sum_{k \in K} (x - x_k + x_k)p_k = \sum_{k \in K} x_kp_k + q_k$$

where

$$q_k := (x - x_k)p_k = (x - x_k)\frac{h_k(x)}{k^4\pi^4} \left[1 + 2k\pi(x - \frac{1}{k\pi})\right] = (x - x_k)\frac{h_k(x)}{k^4\pi^4}. $$
If $f(x)$ is any function defined at the knots $\{x_k \mid k \in K\}$, then the quantum hermite interpolation function $g(x)$ is defined by

$$g(x) = \sum_{k \in \mathbb{Z}^k} f(x_k)p_k + f'(x_k)q_k.$$  

The figure below shows the graph of both the function $f(x) = 1/x$ and the quantum Hermite interpolation polynomial $g(x)$ for $f(x)$ where $h(x) = \sin^2(1/x)$.

![Figure 5: Hermite interpolation $g(x)$ for the function $f(x)$.](image)

References


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