A UNIPODAL ALGEBRA PACKAGE FOR MATHEMATICA

Garret Sobczyk*

Universidad de las Americas
Departamento de Fisico-Matematicas
Apartado Postal #100, Santa Catarina Mártir
72820 Puebla, Pue., México
sobczyk@udlapvms.pue.udlap.mx
FAX: (52-22) 29-2066

September 14, 1995

*I gratefully acknowledge the support given by INIP of the Universidad de las Americas, and CONACYT (The Mexican National Council for Science and Technology) grant 3803-E.

1 Introduction

The Mathematica Package UALGEBRA.M is designed to work in the unipodal number system. This number system $\mathbb{U} = \mathbb{C}[u]$ is the complex number system $\mathbb{C}$ extended to include the *unipotent* u, where $u^2 = 1$ but $u \neq \pm 1$, and is one of the few examples of a *commutative* Clifford algebra. The package extends entire functions of a complex variable to entire functions of a unipodal variable. This package also contains formulas for the solutions of the classical cubic equation [**?**].

A separate package of tools VDET.M is included which is useful in solving the *structure equation* of an arbitrary Clifford number once its minimal polynomial is known.

2 The Unipodal Number System

In the *standard basis* $\{1, u\}$ a *unipodal number* $w \in \mathbb{U}$ has the form $w = w_0 + uw_1$ where $u^2 = 1$ but $u \neq \pm 1$, and $w_0, w_1 \in \mathbb{C}$. The *idempotent basis* $\{u_+, u_-\}$ is defined by

$$u_+ = \frac{1}{2}(1 + u) \quad \text{and} \quad u_- = \frac{1}{2}(1 - u),$$

and satisfies $u_+ + u_- = 1$ and $u_+ - u_- = u$. We say that u_+ and u_- are *idempotents* because $u_+^2 = u_+$ and $u_-^2 = u_-$, and they are *mutually annihilating* because $u_+ u_- = 0$. Using the projective properties of the idempotent basis we can write

$$w = w(u_+ + u_-) = w_+ u_+ + w_- u_-, \tag{1}$$

where $w_+ = w_0 + w_1$ and $w_- = w_0 - w_1$. Conversely, given $w = w_+ u_+ + w_- u_-$ for $w_+, w_- \in \mathbb{C}$, we can recover the coordinates with respect to the standard basis by

$$w_0 = \frac{1}{2}(w_+ + w_-) \quad \text{and} \quad w_1 = \frac{1}{2}(w_+ - w_-). \tag{2}$$

The special properties of the idempotent basis make calculations particularly simple. For example, the binomial theorem takes the simple form

$$(w_+u_+ + w_-u_-)^k = (w_+)^k u_+^k + (w_-)^k u_-^k = (w_+)^k u_+ + (w_-)^k u_- \quad (3)$$

This formula is valid for *all* real numbers $k \in I\!R$, and not just the positive integers. For example, for $k = -1$ we find that

$$1/w = w^{-1} = (1/w_+)u_+ + (1/w_-)u_-,$$

a valid formula for the inverse of $w \in I\!U$, provided that $w_+w_- \neq 0$.

Indeed, the validity of (3) allows us to extend the definitions of *all* of the elementary functions in the complex plane to the corresponding elementary functions in the unipodal plane. If $f(w)$ is such a function for $w = w_+u_+ + w_-u_-$, we define

$$f(w) \equiv f(w_+)u_+ + f(w_-)u_- \quad (4)$$

provided that $f(w_+)$ and $f(w_-)$ are defined.

For example, using the Mathematica Package UALGEBRA.M, we can find $log(2 + 3u)$ by entering the line

$$log[2 + 3u]$$

getting the answer $w = 1/2 \log 5 + \pi i/2 + u(1/2 \log 5 - \pi i/2)$. As a check, we enter the line

$$exp[w]$$

to get back the original $2+3u$. We can equally well find the sybolic expression for $\log(x0 + x1u)$.

The unipodal numbers have been studied in [**?**]. They are used in the next section to find the solutions of the cubic equation.

3 Solutions to Cubic Equation

Historically, the complex numbers were grudgingly accepted because of their utility in solving the cubic equation. We demonstrate here the usefulness of unipodal numbers by finding a formula for the solutions of the venerated reduced cubic equation

$$x^3 + 3ax + b = 0. \tag{5}$$

(The coefficient of the x^2 term of the more general cubic $x^3+p_2x^2+p_1x+p_0=0$, can be eliminated by making the linear substitution $x \longrightarrow x - \frac{1}{3}p_2$.)

The basic unipodal equation $w^n = r$ can easily be solved using the idempotent basis, with the help of equation (3). Writing $w = w_+u_+ + w_-u_-$, and $r = r_+u_+ + r_-u_-$ we get

$$w^n = w_+^n u_+ + w_-^n u_- = r_+u_+ + r_-u_-, \tag{6}$$

so $w_+^n = r_+$ and $w_-^n = r_-$. It follows that $w_+ = r_+^{\frac{1}{n}}\alpha^j$ and $w_- = r_-^{\frac{1}{n}}\alpha^k$ for some integers $0 \le j, k \le n-1$, where α is a primitive n^{th} root of unity and $r_+^{\frac{1}{n}}$ and $r_-^{\frac{1}{n}}$ are arbitrary but fixed n^{th}-roots of r_+ and r_-, respectively. This proves the following theorem.

Theorem 1 *For any positive integer n, the unipodal equation $w^n = r$ has n^2 solutions $w = \alpha^j r_+^{\frac{1}{n}} u_+ + \alpha^k r_-^{\frac{1}{n}} u_-$ for $j, k = 0, 1, \ldots, n-1$, where $\alpha \equiv \exp(2\pi i/n)$.*

The number of roots to the equation $w^n = r$ can be reduced by adding constraints. The following corollary follows immediately from the theorem, by noting that $w_+w_- = \rho \neq 0$ is equivalent to $w_- = \rho/w_+$.

Corollary. The unipodal equation $w^n = r$, subject to the constraint $w_+w_- = \rho$, for a nonzero complex number ρ, has the n solutions

$$w = \alpha^j r_+^{\frac{1}{n}} u_+ + \frac{\rho}{\alpha^j r_+^{\frac{1}{n}}} u_-,$$

for $j = 0, 1, \ldots, n-1$, where $\alpha \equiv \exp(2\pi i/n)$, and $r_+^{\frac{1}{n}}$ denotes any n^{th} root of the complex number r_+.

We are now prepared to solve the reduced cubic equation (5), [**?**].

Theorem 2 *The reduced cubic equation $x^3 + 3ax + b = 0$, has the solutions, for $j = 0, 1, 2$,*

$$x = \frac{1}{2}\left(\alpha^j \sqrt[3]{s+t} + \frac{\rho}{\alpha^j \sqrt[3]{s+t}}\right), \tag{7}$$

where $\alpha = \exp(2\pi i/3) = -\frac{1}{2} + \frac{1}{2}i\sqrt{3}$ *is a primitive cube root of unity, and* $\rho = -4a,\ s = -4b,$ *and* $t = \sqrt{s^2 - \rho^3} = 4\sqrt{b^2 + 4a^3}$.

Proof. The unipodal equation $w^3 = r$, where $r = s + ut$, is equivalent in the standard basis to $(x+yu)^3 = s+tu$, or $(x^3+3xy^2)+u(y^3+3x^2y) = s+ut$. Equating the complex scalar parts gives

$$x^3 + 3xy^2 - s = 0. \tag{8}$$

Making the additional constraint that $w_+w_- = x^2 - y^2 = \rho$, we can eliminate y^2 from (8), getting the equivalent equation

$$x^3 - \frac{3}{4}\rho x - \frac{1}{4}s = 0. \tag{9}$$

The constraint $w_+w_- = \rho$ further implies that

$$\rho^3 = (w_+w_-)^3 = w_+^3 w_-^3 = r_+r_- = s^2 - t^2,$$

which gives $t = \sqrt{s^2 - \rho^3}$. By letting $\rho = -4a$, and $s = -4b$, so $t = \sqrt{s^2 - \rho^3} = 4\sqrt{b^2 + 4a^3}$, the equation (9) becomes the reduced cubic equation $x^3+3ax+b = 0$. Since $r_+ = s+t$, the desired solution (7) is then obtained by taking the complex scalar part of the solution given in the corollary. Q.E.D.

The Mathematica Package UALGEBRA.M has a built in routine to solve the cubic equation $x^3+p_2x^2+p_1x+p_0 = 0$ based on Theorem 2. For example, to solve the cubic equation $x^3+x^2+2x+3 = 0$, open the package "ualgebra" in MATHEMATICA by typing <<ualgebra.m and then enter the line

$$p2 = 1; p1 = 2; p0 = 3;$$

The roots are obtained by entering

$$roots$$

These roots may be checked by entering

$$f[roots]$$

4 Solutions of the Structure Equation

Let $\mathbb{C}_n$ denote the *complex Clifford algebra* of an n-dimensional complex vector space $\mathbb{C}^n$. Let $a \in \mathbb{C}_n$ denote an arbitrary element of this Clifford algebra. By the *minimal polynomial* of a we mean the unique monic polynomial $\psi(\alpha)$ of *least* degree,

$$\psi(\alpha) \equiv \prod_{i=1}^{r} (\alpha - \alpha_i)^{m_i}$$

for the distinct complex numbers $\alpha_1, \alpha_2, \ldots, \alpha_r \in \mathbb{C}$, for which $\psi(a) = 0$. For example, in the complex Clifford algebra $\mathbb{C}_2$ generated by the orthonormal unit vectors e_1, e_2, a general element has the form $a = \alpha_0 + \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_1 e_2$. Noting that

$$(a - \alpha_0)^2 = (\alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_1 e_2)^2 = \alpha_1^2 + \alpha_2^2 - \alpha_3^2 = z^2,$$

we can conclude that a will always satisfy the polynomial equation

$$(\alpha - \alpha_0 - z)(\alpha - \alpha_0 + z) = 0.$$

It follows that the minimal polynomial of an arbitrary element $a \in \mathbb{C}_2$ has degree ≤ 2.

The *Structure equation* of $a \in \mathbb{C}_n$ with the minimal polynomial $\psi(\alpha)$ given above is defined by

$$a = \sum_{i=1}^{r} (\alpha_i + q_i) p_i, \tag{10}$$

where p_i and q_j satisfy the rules

$$\{p_1 + \ldots + p_r = 1,\ p_i p_j = \delta_{ij} p_i,\ q_k^{m_k - 1} \neq 0 \text{ but } q_k^{m_k} = 0,\ p_k q_k = q_k\}. \tag{11}$$

The p_i's are *mutually annihilating idempotents* which partition unity and the q_j's are *nilpotents* with the respective *indexes* m_j's. The commutative algebra $\mathbb{C}\{m_1, m_2, \ldots, m_r\}$ generated by the set $\{p_i, q_i\}$ is isomorphic to the factor ring $\mathbb{C}[\alpha]/ < \psi(\alpha) >$ of the principle ideal $< \psi(\alpha) >$, [?].

The $s \times s$ dimensional *Vandermonde determinant* V, for $s = \sum_{j=1}^{r} m_j$, can be efficiently defined in terms of the column function

$$f_x \equiv f(x) = \{1, x, x^2, \ldots, x^{s-1}\}^T.$$

We have

$$V \equiv \det\{f(\alpha_1), f(\alpha_2), \ldots, f(\alpha_s)\} = \prod_{1\le i<j\le s} (\alpha_j - \alpha_i), \tag{12}$$

where the *normalized* derivatives $f^{(k)}(\alpha)$ are given by

$$f_\alpha^{(k)} \equiv f^{(k)}(\alpha) = \frac{1}{k!} D_\alpha f(\alpha).$$

The *confluent* or *generalized Vandermonde matrix* W is defined by

$$W \equiv \{\underbrace{f_1 f_1^{(1)} \ldots f_1^{(m_1-1)}}_{m_1\ columns}\ \underbrace{f_2 f_2^{(1)} \ldots f_1^{(m_2-1)}}_{m_2\ columns}\ \cdots\ \underbrace{f_r f_r^{(1)} \ldots f_r^{(m_r-1)}}_{m_r\ columns}\}, \tag{13}$$

and has the determinant

$$\det W = \prod_{1\le i<j\le r} (\alpha_j - \alpha_i)^{m_i m_j}. \tag{14}$$

We can now state the *structure theorem* for the element a, [**?**].

Theorem 3 *If $\psi(\alpha)$ is the minimal polynomial of the Clifford number $a \in \mathbb{C}_n$, then a can be expressed in the eigenprojector form*

$$a = \sum_{j=1}^{r} (\alpha_j + q_j) p_j,$$

where the idempotents and nilpotents p_i and q_i, satisfying (11), are the unique polynomials in a of degree $< s = m_1 + \ldots m_r$, specified by

$$p_i = \frac{\det\{\overbrace{f_1 \ldots f_1^{(m_1-1)}}^{m_1} \ldots \overbrace{f_a f_i^{(1)} f_i^{(2)} \ldots f_i^{(m_1-1)}}^{m_i} \ldots \overbrace{f_r \ldots f_r^{(m_r-1)}}^{m_r}\}}{\det W},$$

and

$$q_i = \frac{\det\{\overbrace{f_1 \ldots f_1^{(m_1-1)}}^{m_1} \ldots \overbrace{f_i f_a f_i^{(2)} \ldots f_i^{(m_1-1)}}^{m_i} \ldots \overbrace{f_r \ldots f_r^{(m_r-1)}}^{m_r}\}}{\det W},$$

where the $\det W$ is given in (14).

As an application of this theorem, consider any Clifford number $a \in \mathcal{C}_n$ with the minimal polynomial $\psi(\alpha) = (\alpha - \alpha_1)(\alpha - \alpha_2)^2$. The structure equation of this element is

$$a = \alpha_1 p_1 + (\alpha_2 + q_2)p_2, \tag{15}$$

with the unique solution, for $f(\alpha) = (1, \alpha, \alpha^2)^T$, given by

$$p_1 = \frac{\det\{f_a, f_2, f_2'\}}{\det W} = \frac{D_{\alpha_3} V|_{\alpha_1 \to a, \alpha_3 \to \alpha_2}}{\det W} = \frac{(\alpha_2 - a)^2}{(\alpha_2 - \alpha_1)^2}, \quad q_1 \equiv 0,$$

$$p_2 = \frac{\det\{f_1, f_a, f_2'\}}{\det W} = \frac{D_{\alpha_3} V|_{\alpha_2 \to a, \alpha_3 \to \alpha_2}}{\det W} = \frac{(\alpha_1 - a)(a + \alpha_1 - 2\alpha_2)}{(\alpha_2 - \alpha_1)^2}$$

$$q_2 = \frac{\det\{f_1, f_2, f_a\}}{\det W} = \frac{V|_{\alpha_3 \to a}}{\det W} = \frac{(\alpha_1 - a)(\alpha_2 - a)}{(\alpha_2 - \alpha_1)},$$

where the Vandermonde determinant of this system is $V = (\alpha_3 - \alpha_2)(\alpha_3 - \alpha_1)(\alpha_2 - \alpha_1)$, and

$$\det W = D_{\alpha_3} V|_{\alpha_3 \to \alpha_2} = (\alpha_2 - \alpha_1)^2.$$

The calculations for p_1, p_2, and q_2 can be carried out with the help of the MATHEMATICA PACKAGE VDET.M, as is included as an example at the end of this package.

An important application of eigenprojector form is evaluating any function of a Clifford number. For our example, the *inverse* of a is

$$a^{-1} = \frac{1}{\alpha_1} p_1 + \frac{1}{\alpha_2 + q_2} p_2 = \frac{1}{\alpha_1} p_1 + \frac{\alpha_2 - q_2}{\alpha_2^2} p_2,$$

and just as easily we find

$$\log(a) = \log(\alpha_1) p_1 + \log(\alpha_2 + q_2) p_2 = \log(\alpha_1) p_1 + [\log(\alpha_2) + \frac{q_2}{\alpha_2}] p_2.$$

More generally, let $g(z)$ be any function which has a convergent Taylor series around each of the eigenvalues $z = \alpha_j \in \mathcal{C}$ of the Clifford number a given in (10). Then we can define $g(a)$ by

$$g(a) = g(\sum_{i=1}^{r}(\alpha_i + q_i)p_i) = \sum_{i=1}^{r} g(\alpha_i + q_i)p_i, \tag{16}$$

where

$$g(\alpha_i + q_i) = g(\alpha_i) + g^{(1)}(\alpha_i)q_i + \ldots + \frac{1}{(m_i-1)!}g^{(m_i-1)}(\alpha_i)q_i^{m_i-1}$$

is the Taylor series expansion of $g(z)$ around $z = \alpha_i$. We say that (10) determines the *analytic structure* of a, because $g(a)$ is well defined for any function g which is analytic on the spectrum of a.

The relationship between Clifford algebra and algebras of endomorphisms is discussed in [**?**].

References

[1] D. Hestenes, P. Reany, G. Sobczyk, Unipodal Algebra and Roots of Polynomials, *Advances in Applied Clifford Algebras*, Vol. 1, No. 1, (1991) 31-51.

[2] G. Sobczyk, Hyperbolic Number Plane, *The College Mathematics Journal*, September 1995.

[3] G. E. Sobczyk 1993, Jordan Form in Clifford Algebras, *Clifford Algebras and their Applications in Mathematical Physics*, Proceedings of the Third International Clifford Algebras Workshop, Edited by Fred Brackx, Richard Delanghe, and Herman Serras, Kluwer, Dordrecht, 1993.

[4] G. E. Sobczyk, Structure of Factor Algebras and Clifford Algebra, *Journal of Linear Algebra and Applications*, to appear.