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GENERALIZED VANDERMONDE DETERMINANTS AND APPLICATIONS

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Abstract. Vandermonde determinants are well known and have many applications. Less well known are the *generalized Vandermonde determinants* and their relationship to Hermite interpolation polynomials, factor rings of polynomials, and the spectral decomposition of a linear operator.

1. Generalized Vandermonde Determinants

Vandermonde determinants arise in diverse areas of mathematics. But the familiar Vandermonde determinant is too restrictive. We give formulas for evaluating a more general class of related determinants. Let $\mathbb{K}$ be an arbitrary field. An $m \times m$ *Vandermonde matrix* over $\mathbb{K}$ has the form

$$V = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ x_1 & x_2 & x_3 & \dots & x_m \\ x_1^2 & x_2^2 & x_3^2 & \dots & x_m^2 \\ x_1^3 & x_2^3 & x_3^3 & \dots & x_m^3 \\ \dots & \dots & \dots & \dots & \dots \\ x_1^{m-1} & x_2^{m-1} & x_3^{m-1} & \dots & x_m^{m-1} \end{pmatrix}$$

The Vandermonde matrix V can be efficiently defined in terms of the column vector function $X(x) = \{1, x, x^2, \dots, x^{m-1}\}^T$ evaluated at the points $x = x_1, x_2, \dots, x_m \in \mathbb{K}$; we write $V = \{X(x_1), X(x_2), \dots, X(x_m)\}$. The *determinant* of V is given by the formula

$$\det V = \prod_{1 \le i < j \le m} (x_j - x_i) \tag{1}$$

whose proof is usually left as an exercise in using elementary row operations.

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We now focus attention upon other Vandermonde determinants that are only infrequently mentioned but which give insight into the relationship between a polynomial and its roots. These determinants are best defined in terms of the *normalized derivatives* $X^{[k]}(x)$ of the column vector function $X(x)$,

$$X^{[k]}(x) := \frac{1}{k!}\partial_x^k X(x),$$

[6, p.60]. For example,

$$X^{[1]}(x) = \{0, 1, \binom{2}{1} x, \binom{3}{1} x^2, \ldots, \binom{m-1}{1} x^{m-2}\}^T,$$

and

$$X^{[2]}(x) = \{0, 0, 1, \binom{3}{2} x, \ldots, \binom{m-1}{2} x^{m-3}\}^T,$$

so that $\{X, X^{[1]}, \ldots, X^{[k]}, \ldots, X^{[m-1]}\} =$

$$\begin{pmatrix} 1 & 0 & \ldots & 0 & \ldots & 0 \\ x & 1 & \ldots & 0 & \ldots & 0 \\ . & . & \ldots & . & \ldots & . \\ . & . & \ldots & . & \ldots & . \\ x^{m-1} & \binom{m-1}{1} x^{m-2} & \ldots & \binom{m-1}{k} x^{m-k-1} & \ldots & 1 \end{pmatrix}.$$

It follows that

$$\{X(0), X^{[1]}(0), \ldots, X^{[k]}(0), \ldots, X^{[m-1]}(0)\} = I \tag{2}$$

where I is the identity $(m \times m)$-matrix; an important property which will be used later. Note that this definition remains valid even in the case of the finite fields $\mathbb{F}_p$, if we merely replace the coefficient by 0 whenever a factor of p appears. We use the shortened notation $X_j^{[k]} := X^{[k]}(x_j)$, and $X_y^{[k]} := X^{[k]}(y)$ for the column vector function $X^{[k]}(x)$ evaluated at the points $x = x_j$ and at the indeterminate $x = y$, respectively.

With the above notation in hand, we are able to prove simple formulas for evaluating the determinant of the *generalized Vandermonde matrix*

$$W := \{\underbrace{X_1 X_1^{[1]} \ldots X_1^{[m_1-1]}}_{m_1 \ columns} \underbrace{X_2 X_2^{[1]} \ldots X_2^{[m_2-1]}}_{m_2 \ columns} \ldots \underbrace{X_r X_r^{[1]} \ldots X_r^{[m_r-1]}}_{m_r \ columns}\}, \tag{3}$$

where $m = m_1 + m_2 + \ldots + m_r$. Later, we give a formula for evaluating the determinant of the closely related matrix obtained by replacing any single column of the above matrix by the column $X_x := X(x)$.

The proof of the following theorem utilizes the concept of the derivative of a determinant. For example, if we partial differentiate with respect to x_3 the left and right sides of the formula

$$\det\{X_1, X_2, X_3\} = (x_2 - x_1)(x_3 - x_1)(x_3 - x_2),$$

which is the Vandermonde determinant (1) with $m = 3$, and evaluate it at $x_3 = x_2$, we get

$$\partial_3 \det\{X_1, X_2, X_3\}|_{x_3=x_2} = \partial_3 (x_2 - x_1)(x_3 - x_1)(x_3 - x_2)|_{x_3=x_2}$$

or

$$\det\{X_1, X_2, X_2^{[1]}\} = (x_2 - x_1)^2.$$

Explicitly, we have shown that

$$\det \begin{pmatrix} 1 & 1 & 0 \\ x_1 & x_2 & 1 \\ x_1^2 & x_2^2 & 2x_2 \end{pmatrix} = (x_2 - x_1)^2.$$

More generally, we have the result

Theorem 1. *If* $m = m_1 + m_2 + \ldots + m_r$ *and*

$$W = \{\underbrace{X_1 X_1^{[1]} \ldots X_1^{[m_1-1]}}_{m_1 \ columns} \ \underbrace{X_2 X_2^{[1]} \ldots X_2^{[m_2-1]}}_{m_2 \ columns} \ \ldots \ \underbrace{X_r X_r^{[1]} \ldots X_r^{[m_r-1]}}_{m_r \ columns}\},$$

then $\det W = \prod_{1 \le i < j \le r} (x_j - x_i)^{m_i m_j}$.

Proof. We can assume without loss of generality that $1 \le m_1 \le \ldots \le m_r$. We argue by reverse induction on the number of blocks $r \le m$ for each fixed $m > 0$. For $r = m$, we have $m_1 = m_2 = \ldots = m_r = 1$, and W is just the classical Vandermonde matrix with determinant as in (1). We now assume true for r blocks where $0 \le m - r \le k$, and prove true for any matrix W which has r blocks which satisfy $0 \le m - r = k + 1$. Thus, we must evaluate

$$W = \{\underbrace{X_1 X_1^{[1]} \ldots X_1^{[m_1-1]}}_{m_1 \ columns} \ \underbrace{X_2 X_2^{[1]} \ldots X_2^{[m_2-1]}}_{m_2 \ columns} \ \ldots \ \underbrace{X_r X_r^{[1]} \ldots X_r^{[m_r-1]}}_{m_r \ columns}\},$$

where $0 \le m - r = k + 1$ by assuming true for any number of blocks s such that $0 \le m - s \le k$.

Since $1 \le m_1 \le m_2 \le \ldots \le m_r$, it follows that $m_r \ge 2$. Let

$$W_x := \{\underbrace{X_1 X_1^{[1]} \ldots X_1^{[m_1-1]}}_{m_1 \ columns} \ \underbrace{X_2 X_2^{[1]} \ldots X_2^{[m_2-1]}}_{m_2 \ columns} \ \ldots \ \underbrace{X_r X_r^{[1]} \ldots X_r^{[m_r-2]} X_x}_{m_r \ columns}\},$$

for the indeterminate x. Then W_x contains $s = r + 1$ blocks, so by the induction hypothesis, $\det W_x =$

$$\prod_{1 \le i < j \le r-1} (x_j - x_i)^{m_i m_j} \prod_{1 \le i < r} (x_r - x_i)^{(m_r-1)m_i} \prod_{1 \le i < r} (x - x_i)^{m_i} (x - x_r)^{m_r-1}.$$

Applying the operator $\frac{1}{(m_r-1)!}\partial_x^{m_r-1}$ to both sides of this equation, and evaluating at $x = x_r$ gives the desired result

$$\det W = \frac{1}{(m_r-1)!}[\partial_x^{m_r-1}\det W_x]_{x=x_r}$$

$$= \prod_{1\le i<j\le r-1}(x_j-x_i)^{m_i m_j}\prod_{1\le i<r}(x_r-x_i)^{(m_r-1)m_i}\prod_{1\le i<r}(x_r-x_i)^{m_i}$$

$$= \prod_{1\le i<j\le r}(x_j-x_i)^{m_i m_j}.$$

□

In the proof of the above theorem, we have not been careful about the case when the field $\mathbb{K}$ is a finite field. Nevertheless, the theorem remains valid even in this case.

There is another class of determinants that can be efficiently evaluated by algebraic formulas. Replacing the k^{th} column in the i^{th} block of W with the column vector $X_x = X(x)$ we get the matrix $W_{ik} :=$

$$\{\underbrace{X_1X_1^{[1]}\dots X_1^{[m_1-1]}}_{m_1\ columns}\ \cdots\ \underbrace{X_i\dots X_i^{[k-2]}X_xX_i^{[k]}\dots X_i^{[m_i-1]}}_{m_i\ columns}\ \cdots\ \underbrace{X_rX_r^{[1]}\dots X_r^{[m_r-1]}}_{m_r\ columns}\}.$$

The determinant of W_{ik} is given by the formula

$$\det W_{ik} = q_i^{k-1}(x)\ \det W \tag{4}$$

where $q_i^k(x)$ is defined for $k\ge 0$ by

$$q_i^k(x) := \Big[\frac{1}{\psi_i(x_i)} + \Big(\frac{1}{\psi_i(x)}\Big)^{[1]}\Big|_{x=x_i}(x-x_i) + \Big(\frac{1}{\psi_i(x)}\Big)^{[2]}\Big|_{x=x_i}(x-x_i)^2 +$$

$$+\dots+\Big(\frac{1}{\psi_i(x)}\Big)^{[m_i-k-1]}\Big|_{x=x_i}(x-x_i)^{m_i-k-1}\Big](x-x_i)^k\psi_i(x)$$

for $\psi_i(x) = \prod_{j\ne i}(x-x_j)^{m_j}$. Note that $q_i^k(x)$ is just the first $m_i - k$ terms of the Taylor series expansion of the rational function $1/\psi_i(x)$ around the point $x = x_i$.

For example, suppose we wish to evaluate the determinant

$$\det\begin{vmatrix} 1 & 1 & 1 & 0 \\ x_1 & x_2 & x & 0 \\ x_1^2 & x_2^2 & x^2 & 1 \\ x_1^3 & x_2^3 & x^3 & 3x_2 \end{vmatrix}$$

It follows that $r = 2$, $m_1 = 1$ and $m_2 = 3$, so that $\det W = (x_2 - x_1)^3$ and

$$q_2^1(x) = \Big[\frac{1}{x_2-x_1} - \frac{(x-x_2)}{(x_2-x_1)^2}\Big](x-x_2)(x-x_1).$$

Thus we have

$$\det W_{22} = q_2^1(x)\ \det W = (x_1 - x_2)(x - x_1)(x - x_2)(x + x_1 - 2x_2).$$

The reader might like to try his hand at a proof of the general formula (4), or he might consult the reference [4, p.176].

2. Spectral Basis

Let $\mathbb{R}[x]_h := \mathbb{R}[x]/h(x)$ be the *factor ring* of polynomials in the indeterminate x generated by the principal ideal of the polynomial $h(x) = \prod_{i=1}^{r}(x - x_i)^{m_i}$, [2]. The factor ring $\mathbb{R}[x]_h$, as an m-dimensional vector space, is spanned by the *standard basis*

$$X(x) = \{1, x, x^2, \cdots, x^{m-1}\}^T, \tag{5}$$

which we write in column vector form. Using this notation, any polynomial $g(x) \in \mathbb{R}[x]_h$ has the form $g(x) = \sum_{i=0}^{m-1} a_i x^i = A_X X(x)$ where A_X is the *row vector*

$$A_X := \begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_{m-1} \end{pmatrix}$$

of components of $g(x)$ in the standard basis X.

The *spectral basis* of the factor ring $\mathbb{R}[x]_h$ is the column vector of m polynomials

$$S(x) := \begin{pmatrix} p_1 & q_1 & \cdots & q_1^{m_1-1} & \ldots & p_r & q_r & \cdots & q_r^{m_r-1} \end{pmatrix}^T \tag{6}$$

in $\mathbb{R}[x]_h$ defined by the equation

$$S(x) := W^{-1}X(x), \tag{7}$$

where W is the invertible generalized Vandermonde matrix (3). The polynomials of the spectral basis are also specified directly in (4), which is a form of *Cramer's rule* for $i = 1, \ldots, r$, and $k = 0, 1, \ldots, m_i - 1$, where $p_i := q_i^0$. The polynomial $g(x) \in \mathbb{R}[x]_h$, defined above, is easily expressed in the spectral basis $S(x)$,

$$g(x) = A_X X(x) = A_X W W^{-1} X(x) = A_S S(x)$$

where $A_S := A_X W$ is the row vector of components of $g(x)$ in the spectral basis. Expressing $g(x)$ in the spectral basis is equivalent to the famous *Chinese Remainder Theorem* for the modular number system Z_h modulo the composite number $h \in \mathbb{N}$, [3, p. 337].

The spectral basis of the factor ring $\mathbb{R}[x]_h$ has great utility because the polynomials of this basis enjoy the following simple *rules of multiplication:*

- $\sum_{i=1}^{r} p_i = 1, \quad p_i p_j = \delta_{ij} p_i, \quad p_i q_i = q_i$
- $q_i^{m_i-1} \neq 0, \quad q_i^{m_i} = 0,$

for all $i, j = 1, \ldots, r$. Consequently, $p_i q_j = p_i p_j q_j = \delta_{ij} q_j$ and $q_i^a q_j^b = q_i^a p_i q_j^b = \delta_{ij} q_j^{a+b}$. Because of these properties, the p_i form a *complete system of (primitive)* *idempotents*, and the *nilpotents* q_i have *index of nilpotency* m_i for $i = 1, \ldots, r$, respectively. The algebraic properties of the spectral basis are most easily derived as a consequence of the *Euclidean algorithm*, [3].

For example, let $h(x) = (x - x_1)(x - x_2)^3$, and $X = \{1, x, x^2, x^3\}^T$. Then

$$W = \begin{pmatrix} X(x_1) & X(x_2) & X^{[1]}(x_2) & X^{[2]}(x_2) \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & 0 & 0 \\ x_1 & x_2 & 1 & 0 \\ x_1^2 & x_2^2 & 2x_2 & 1 \\ x_1^3 & x_2^3 & 3x_2^2 & 3x_2 \end{pmatrix}.$$

The spectral basis for $\mathbb{R}[x]_h$ is given by

$$S(x) = W^{-1}X(x).$$

For $x_1 = 2$ and $x_2 = 3$, we find

$$S(x) = \begin{pmatrix} p_1 \\ p_2 \\ q_2 \\ q_2^2 \end{pmatrix} = \begin{pmatrix} 27 - 27x + 9x^2 - x^3 \\ -26 + 27x - 9x^2 + x^3 \\ 24 - 26x + 9x^2 - x^3 \\ -18 + 21x - 8x^2 + x^3 \end{pmatrix}. \tag{8}$$

We also have $X(x) = WS(x)$, or

$$X(x) = \begin{pmatrix} 1 \\ x \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} p_1 + p_2 \\ 2p_1 + 3p_2 + q_2 \\ 2^2p_1 + 3^2p_2 + 6q_2 + q_2^2 \\ 2^3p_1 + 3^3p_2 + 27q_2 + 9q_2^2 \end{pmatrix}. \tag{9}$$

The relationship

$$x = 2p_1 + 3p_2 + q_2 \tag{10}$$

is called the *spectral decomposition* of x in $\mathbb{R}[x]_{(x-2)(x-3)^3}$. Note the spectral decompositions for x^2 and x^3 follow by *squaring* and *cubing* both sides of the expression (10) for the spectral decomposition of x, using the rules of multiplication given above.

3. Hermite Interpolation

An easy consequence of (2) and (7) is that

$$W^{-1} = \{S(0), S^{[1]}(0), \ldots, S^{[m-1]}(0)\}$$

and, using (3) and (7),

$$I = \{S(x_1), \ldots, S^{[m_1-1]}(x_1); \ldots; S(x_r), \ldots, S^{[m_r-1]}(x_r)\}.$$

It follows that $S(x)$ is a basis of orthogonal *Hermite Polynomials*, [1]. Just as the *truncated Taylor series* of an analytic function $f(x)$ can be obtained by taking the remainder of the result obtained by dividing the infinite series expansion of $f(x)$

by $(x - x_1)^m$, the *Hermite interpolation polynomial* of $f(x)$ is obtained by taking the remainder obtained by dividing $f(x)$ by the more general polynomial $h(x) = \prod_{i=1}^{r}(x - x_i)^{m_i}$.

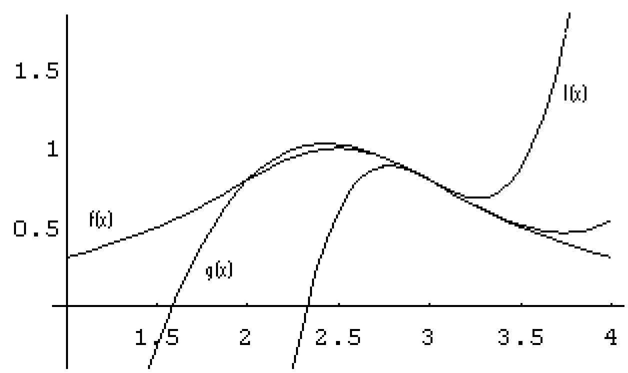

FIGURE 1. Hermite interpolation $g(x)$ and truncated Taylor series $l(x)$, for the function $f(x)$. Note that Hermite interpolation forces agreement at both the points $x = 2$ and $x = 3$.

As an example, we shall use the spectral decomposition (10) of x to find the Hermite interpolation polynomial $g(x) \in \mathbb{R}[x]_{(x-2)(x-3)^3}$ of the rational function $f(x) = \frac{4}{4+(2x-5)^2}$. Using (10) and the rules for multiplication of the spectral basis, we find that

$$g(x) := f(2p_1 + (3 + q_2)p_2) = f(2)p_1 + f(3 + q_2)p_2$$

$$= f(2)p_1 + [f(3) + f^{[1]}(3)q_2 + f^{[2]}(3)q_2^2]p_2,$$

$$= f(2)p_1 + [f(3) + f^{[1]}(3)(x-3) + f^{[2]}(3)(x-3)^2]p_2$$

where $f^{[k]}(x) = \frac{1}{k!}\frac{d^k f(x)}{dx^k}$ is the normalized k^{th} derivative of $f(x)$. The last equality above shows that the Hermite interpolation polynomial $g(x)$ of $f(x)$ gives a *Taylor series*-like expansion of $f(x)$ around the points $x = 2$ and $x = 3$. The graph above shows $f(x)$, $g(x)$, and $l(x)$ where

$$l(x) = f(3) + f^{[1]}(3)(x-3) + f^{[2]}(3)(x-3)^2 + f^{[3]}(3)(x-3)^3$$

is the first four terms of the Taylor series expansion of $f(x)$ around $x = 3$. The Hermite polynomial of $f(x)$ in the factor ring $\mathbb{R}[x]_h$ for $h(x) = (x-3)^4$, gives exactly the polynomial $l(x)$.

4. Linear algebra

The relationship (7), $X(x) = WS(x)$, between the standard basis $X(x)$ and the spectral basis $S(x)$ of the factor ring $\mathbb{K}[x]/h(x)$ can be applied immediately to a linear operator t to obtain

$$X(t) = WS(t), \tag{11}$$

where

$$h(x) := \det\{x\mathbf{1} - t\} = \prod_{i=1}^{r}(x - x_i)^{n_i}$$

is the *characteristic polynomial* of t and $\mathbf{1}$ is the identity operator. We have already seen examples of this relationship in (8) and (9). If the *minimal polynomial* $\varphi(x)$ of t is known, the spectral basis can be calculated with fewer multiplications, and the algebra thereby generated by the operator t is *isomorphic* to the factor ring $\mathbb{K}[x]/\varphi(x)$, [3, p.343]. Recall that every operator satisfies its characteristic equation (Cayley-Hamilton), and that the minimal polynomial is the polynomial of least degree which has this property. The *spectral decomposition* of the operator t is just the special case of the relationship (11)

$$t = \sum_{i=1}^{r}(x_i + q_i)p_i, \tag{12}$$

where $p_i = p_i(t)$, and $q_i = q_i(t)$.

The spectral decomposition completely determines the *analytic structure* of t, in so far as that, using (12), any function $f(t)$ can immediately be written down:

$$f(t) := f(\sum_{j=1}^{r}(x_j + q_j)p_j) = \sum_{j=1}^{r} f(x_j + q_j)p_j \tag{13}$$

$$= \sum_{j=1}^{r}[f(x_j) + f^{[1]}(x_j)q_j(x) + \ldots + f^{[m_j-1]}(x_j)q_j^{m_j-1}(x)]p_j(t),$$

$$= \sum_{j=1}^{r}[f(x_j) + f^{[1]}(x_j)(t - x_j) + \ldots + f^{[m_j-1]}(x_j)(t - x_j)^{m_j-1}]p_j(t)$$

$$:= \sum_{j=1}^{r} \left[f^{[m_j-1]}(x_j), \ldots, f^{[1]}(x_j), f(x_j)\right]_{t-x_j} p_j(t).$$

Many structure theorems for linear operators can be easily derived from the spectral decomposition (12), [4], [5]. For example, an operator is *diagonal* iff $t = \sum x_i p_i$, and an operator has *cyclic structure* iff its minimal polynomial is identical with its characteristic polynomial. Both of these results follow by choosing vectors v_i such that $p_i v_i = v_i$ and $q_i^{m_i-1} v_i \neq 0$, for $i = 1, \ldots, r$ and then defining $v = \sum v_i$. The set of *Jordan chains* of vectors

$$\{S(t)v\}^T := \{v_1, q_1 v_1, \ldots q_1^{m_1-1} v_1, \ldots, v_r, q_r v_r, \ldots, q_r^{m_r-1} v_r\}$$

make up a *row* of *generalized eigenvectors* with the property that

$$t\{S(t)v\}^T = \{S(t)v\}^T T_J, \tag{14}$$

where T_J is the *Jordan canonical form* of the operator t with respect to the basis $\{S(t)v\}^T$. On the other hand, using (7),

$$\{X(t)v\}^T = \{WS(t)v\}^T$$

gives the change of basis from the spectral basis $\{S(t)v\}^T$ to the *cyclic basis* $\{X(t)v\}^T$ generated by the vector v. In the cyclic basis $\{X(t)v\}^T$,

$$t\{X(t)v\}^T = \{X(t)v\}^T T_X \tag{15}$$

where T_X is the *companion matrix* of t.

For example, both the minimal and characteristic polynomial of the matrix

$$T = \begin{pmatrix} 3 & 10 & -29 & 47 \\ 0 & 4 & -3 & 5 \\ 0 & 4 & -7 & 16 \\ 0 & 2 & -5 & 11 \end{pmatrix}$$

is $h(x) = (x-2)(x-3)^3$, from which it follows that the spectral decomposition of T is given by (10),

$$T = 2P_1 + 3P_2 + Q_2, \tag{16}$$

where, using (8),

$$P_1 = 27\mathbf{1} - 27T + 9T^2 - T^3 = \begin{pmatrix} 0 & -12 & 24 & -36 \\ 0 & -1 & 2 & -3 \\ 0 & -4 & 8 & -12 \\ 0 & -2 & 4 & -6 \end{pmatrix},$$

$$P_2 = \mathbf{1} - P_1 = \begin{pmatrix} 1 & 12 & -24 & 36 \\ 0 & 2 & -2 & 3 \\ 0 & 4 & -7 & 12 \\ 0 & 2 & -4 & 7 \end{pmatrix}, \quad Q_2 = T - 2P_1 - 3P_2 = \begin{pmatrix} 0 & -2 & -5 & 11 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & -2 & 4 \\ 0 & 0 & -1 & 2 \end{pmatrix}$$

and

$$Q_2^2 = \begin{pmatrix} 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Choosing the *third* column vector v_1 of P_1, and the *fourth* column vector v_2 of P_2, and defining $v = v_1 + v_2$, gives

$$v_1 = \begin{pmatrix} 24 \\ 2 \\ 8 \\ 4 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 36 \\ 3 \\ 12 \\ 7 \end{pmatrix}, \quad v = v_1 + v_2 = \begin{pmatrix} 60 \\ 5 \\ 20 \\ 11 \end{pmatrix}.$$

Checking, we find that $P_1 v = v_1$, and $P_2 v = v_2$ and $Q_2^2 v = Q_2^2 v_2 \neq 0$, as required. A set of *generalized eigenvectors* for the matrix T is given by

$$C := \{P_1 v, P_2 v, Q_2 v, Q_2^2 v\} = \begin{pmatrix} 24 & 36 & 11 & -2 \\ 2 & 3 & 2 & 0 \\ 8 & 12 & 4 & 0 \\ 4 & 7 & 2 & 0 \end{pmatrix},$$

and the *cyclic basis* generated by the vector v is

$$V := \{v, Tv, T^2 v, T^3 v\} = \begin{pmatrix} 60 & 167 & 484 & 1443 \\ 5 & 15 & 47 & 151 \\ 20 & 56 & 164 & 496 \\ 11 & 31 & 91 & 275 \end{pmatrix}.$$

The Jordan canonical form of the operator T is then given by

$$T_J = C^{-1} T C = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 3 \end{pmatrix},$$

and

$$V^{-1} T V = \begin{pmatrix} 0 & 0 & 0 & -54 \\ 1 & 0 & 0 & 81 \\ 0 & 1 & 0 & -45 \\ 0 & 0 & 1 & 11 \end{pmatrix}$$

gives the *companion matrix* for the operator t with respect to the cyclic basis V.

Using the spectral decomposition (16) and (13), the inverse of T is given by

$$T^{-1} = \frac{1}{2} P_1 + \frac{1}{3 + Q_2} P_2 = \frac{1}{2} P_1 + \frac{1}{3} P_2 - \frac{1}{9} Q_2 + \frac{1}{27} Q_2^2 = \frac{1}{54} \begin{pmatrix} 18 & -96 & 248 & -394 \\ 0 & 9 & 24 & -39 \\ 0 & -36 & 102 & -132 \\ 0 & -18 & 42 & -48 \end{pmatrix}$$

Similarly, we can calculate

$$\sqrt{T} = \sqrt{2} P_1 + \sqrt{3} P_2 + \frac{1}{2\sqrt{3}} Q_2 - \frac{1}{24\sqrt{3}} Q_2^2.$$

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