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EQUIVALENCE OF AN IDENTITY IN VECTOR ANALYSIS TO QUATERNION ASSOCIATIVITY, AND RAMIFICATIONS

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Real Quaternions. Identifying the unit vectors **i**, **j**, **k** with quaternion imaginaries, each real quaternion **q** has the form $a + \mathbf{A}$, with scalar component a and vector (or imaginary) component **A**. In terms of the usual dot and cross products, the product **AB** of imaginary quaternions is easily seen to be $-\mathbf{A}\cdot\mathbf{B} + \mathbf{A}\times\mathbf{B}$. Thus

$$(a + \mathbf{A})(b + \mathbf{B}) = ab + b\mathbf{A} + a\mathbf{B} - \mathbf{A}\cdot\mathbf{B} + \mathbf{A} + \mathbf{B},$$

and the associator

$$\langle(a + \mathbf{A})(b + \mathbf{B})\rangle(c + \mathbf{C}) - (a + \mathbf{A})\langle(b + \mathbf{B})(c + \mathbf{C})\rangle = 0 + \mathbf{0}$$
$$= (\mathbf{A}\cdot\mathbf{B})\mathbf{C} - (\mathbf{A}\times\mathbf{B})\times\mathbf{C} - (\mathbf{B}\cdot\mathbf{C})\mathbf{A} + \mathbf{A}\times(\mathbf{B}\times\mathbf{C}),$$

or

$$(\mathbf{A}\times\mathbf{B})\times\mathbf{C} - \mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = (\mathbf{A}\cdot\mathbf{B})\mathbf{C} - \mathbf{A}(\mathbf{B}\cdot\mathbf{C}). \qquad (*)$$

See, e.g., [3, (33) and (34), p. 290]. Conversely, identity (*) implies that multiplication of quaternions is associative.

It is well known that the algebra of real 3-vectors with the cross product is the Lie algebra of the real three-dimensional orthogonal group;

$$\text{if } \mathbf{i} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \mathbf{j} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad \mathbf{k} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

then the Lie algebra also may be described as the algebra of 3 by 3 real antisymmetric matrices $A, B, \ldots$, with the commutator or Lie bracket $[\mathbf{A},\mathbf{B}] = AB - BA = \mathbf{A}\times\mathbf{B}$ as product. We have $[\mathbf{i},\mathbf{j}] = \mathbf{i}\times\mathbf{j} = \mathbf{k}$, $\mathbf{j}\times\mathbf{k} = \mathbf{i}$, $\mathbf{k}\times\mathbf{i} = \mathbf{j}$.

Change to $\mathbf{j}\times\mathbf{k} = -\mathbf{i}$. The Lie algebra of the real two-dimensional special linear group $SL_2(R)$ is the algebra of all real 2-by-2 matrices having trace 0, with the commutator product. (See, e.g., [4, p. 89].) Choosing as basis

$$\mathbf{i} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \mathbf{j} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{k} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

we have $[\mathbf{i},\mathbf{j}] = \mathbf{k}$, $[\mathbf{j},\mathbf{k}] = -\mathbf{i}$, $[\mathbf{k},\mathbf{i}] = \mathbf{j}$, the same as the cross-product algebra except that $[\mathbf{j},\mathbf{k}] = -\mathbf{i}$ instead of $[\mathbf{j},\mathbf{k}] = +\mathbf{i}$. The effect of the change of sign is to make the analogous algebra to the quaternion algebra not associative: $(\mathbf{ij})\mathbf{k} = \mathbf{k}^2 = -1$ but $\mathbf{i}(\mathbf{jk}) = -\mathbf{i}^2 = +1$. Like the quaternions, this latter algebra is noncommutative; also there are non-null divisors of zero, e.g., $(1 + \mathbf{i} + \mathbf{j} + \mathbf{k})(1 + \mathbf{i} - \mathbf{j} + \mathbf{k}) = 0$ but $(1 + \mathbf{i} - \mathbf{j} + \mathbf{k})(1 + \mathbf{i} + \mathbf{j} + \mathbf{k}) = 4(\mathbf{i} + \mathbf{k})$. See [1], [5].

Vectors with Complex Components. In order to consider complex 3-vectors $\mathbf{A}_1 + i\mathbf{A}_2$,

$\mathbf{B}_1 + i\mathbf{B}_2, \ldots$, where $\mathbf{A}_i$, $\mathbf{B}_i$, etc., are real vectors, to avoid confusion we now change notation from **i**, **j**, **k** to, respectively, e_1, e_2, e_3, freeing i to be $\sqrt{-1}$. For $\mathbf{A} = \mathbf{A}_1 + i\mathbf{A}_2$, $\mathbf{B} = \mathbf{B}_1 + i\mathbf{B}_2$, the product

$$\mathbf{A}\cdot\mathbf{B} = (\mathbf{A}_1 + i\mathbf{A}_2)\cdot(\mathbf{B}_1 + i\mathbf{B}_2) = (\mathbf{A}_1\cdot\mathbf{B}_1 - \mathbf{A}_2\cdot\mathbf{B}_2) + i(\mathbf{A}_2\cdot\mathbf{B}_1 + \mathbf{A}_1\cdot\mathbf{B}_2)$$

is the negative of the scalar part, and $\mathbf{A}\times\mathbf{B} = (\mathbf{A}_1 + i\mathbf{A}_2)\times(\mathbf{B}_1 + i\mathbf{B}_2)$ is the quaternion imaginary part, of the product of imaginary complex quaternions

$$(A_{11} + iA_{21})e_1 + (A_{12} + iA_{22})e_2 + (A_{13} + iA_{23})e_3$$

and

$$(B_{11} + iB_{21})e_1 + (B_{12} + iB_{22})e_2 + (B_{13} + iB_{23})e_3.$$

As in the case of real quaternions, associativity of multiplication of complex quaternions is equivalent to identity (*) in which **A**, **B**, **C** are allowed to be complex 3-vectors. (The complex quaternions are an associative algebra, of which the underlying real linear space is eight-dimensional, and which has divisors of zero, e.g., $(i + e_1)(i - e_1) = i^2 - e_1^2 = -1 + 1 = 0$.) In case of complex 3-vectors with complex orthogonality, there are isotropic vectors, i.e., non-null vectors that are self-orthogonal, e.g., $\mathbf{A}\cdot\mathbf{A} = |\mathbf{A}_1|^2 - |\mathbf{A}_2|^2 + 2i\mathbf{A}_1\cdot\mathbf{A}_2 = 0$ if $|\mathbf{A}_1| = |\mathbf{A}_2|$ with $\mathbf{A}_1\cdot\mathbf{A}_2 = 0$. To make $\underline{\mathbf{A}}\times\underline{\mathbf{B}}$ Hermitian-orthogonal to **A** and to **B**, we may define the cross product by $\mathbf{A}\otimes\mathbf{B} = \bar{\mathbf{A}}\times\bar{\mathbf{B}}$, where $\bar{\mathbf{A}}$ denotes the complex conjugate vector $\mathbf{A}_1 - i\mathbf{A}_2$ of $\mathbf{A} = \mathbf{A}_1 + i\mathbf{A}_2$, and $\times$ is the usual real cross product. The Hermitian inner product is left invariant by unitary transformations. The Lie algebra of the unitary group is the commutator algebra of the anti-Hermitian 3-by-3 matrices, which is not isomorphic with the algebra of complex antisymmetric matrices with $\otimes$ as product; the latter algebra is merely the image under conjugation of the complexification of the algebra of the three-dimensional orthogonal group. If a complex matrix $A = A_1 + iA_2$ is anti-Hermitian, then A_1 must be antisymmetric and A_2 symmetric. Since the product of a real antisymmetric matrix by a complex number is not anti-Hermitian, it is clear that the Lie algebra of the unitary group is only a real Lie algebra; i.e., it is not closed under scalar multiplication by complex scalars. Apparently if there is an extension to complex vectors of the relation between real quaternions and the cross product of real vectors, having the desired property of Hermitian orthogonality, it is not a trivial extension.

Cayley Numbers. Cayley numbers or octonions are a nonassociative real eight-dimensional algebra, without zero divisors, which has as a basis the scalar unit 1 and seven imaginaries $\{e_i\}$, $i = 1, \ldots, 7$. For a description, see, e.g., [2]. Each of the triples $\{e_1,e_2,e_4\}$, $\{e_2,e_3,e_5\}$, $\{e_3,e_4,e_6\}$, $\{e_4,e_5,e_7\}$, $\{e_5,e_6,e_1\}$, $\{e_6,e_7,e_2\}$, $\{e_7,e_1,e_3\}$ is analogous to $\{i,j,k\}$ and, with 1, forms a basis for a real quaternionic subalgebra. The triples are a "Steiner system" that covers all of the pairs $\{e_i,e_j\}$, each exactly once. The multiplication table is $e_i^2 = -1$, $i = 1, \ldots, 7$, with the product of the three imaginaries in each triple also -1, e.g., $e_1e_2e_4 = -1$, which is shorthand for $e_2e_4 = -e_4e_2 = e_1, e_4e_1 = -e_1e_4 = e_2, e_1e_2 = -e_2e_1 = e_4$. The seven triples are associative; all other triples are not, e.g., $e_1(e_2e_5) = e_1(-e_3) = -e_7$, but $(e_1e_2)e_5 = e_4e_5 = +e_7$. The algebra L_7 of seven-dimensional vectors $A = \Sigma_{i=1}^7 a_ie_i = \{A_1, A_2, \ldots, A_7\}$, where $A_1, A_2, \ldots, A_7$ are 3-vectors $A_1 = (a_1,a_2,a_4)$, $A_2 = (a_2,a_3,a_5)$, etc., with product defined by $e_i^2 = 0$,

$$A\times B = A_1\times B_1 + A_2\times B_2 + \cdots + A_7\times B_7,$$

is a Lie algebra that bears a similar relationship to the algebra of Cayley numbers, to that of the real 3-vectors with $\times$ to the quaternions; i.e., the Cayley numbers may be regarded as $a + A$, $b + B$, etc., where a, b, etc., are the scalar, and A, B, etc., the seven-dimensional, purely imaginary components of the octonions.

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