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Partitions Induced by Linear Function-Spaces

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1. *Introduction*

For a set S, by a *partition* let us understand an expression $S = \cup A_i$, over all i in an index set I, where the A_i's are distinct, incomparable by inclusion, but not necessarily disjoint. By a *decomposition* let us understand a partition in which the A_i's are required to be disjoint. An equivalence relation α in S, as is well-known, induces a corresponding decomposition of S such that $s \alpha s'$ if and only if s and s' belong to the same A_i; we refer to the subsets A_i as α-subsets. Let us write $(s, s')_2$ in case s, s' are in the binary relation of belonging to the same subset of a partition, and similarly refer to the sets A_i of the partition as $(\ )_2$-*subsets*. Equivalence is a binary relation; we shall consider a sequence of respectively binary, ternary, quaternary, . . . relations in S, and when sets of elements of S are in those relations, we shall write $(s, s')_2$, $(s, s', s'')_3$, $(s, s', s'', s''')_4$, ... Maximal subsets of mutually $(\ )_3$-, $(\ )_4$-, . . . related elements will be called $(\ )_3$-*subsets*, $(\ )_4$-*subsets*, ...

After discussion of the cases n = 1 through n = 3, the $(\ )_n$-subsets are shown to be constituted of the elements which are involved in the subsets of an equivalence decomposition of a certain subset of a Cartesian product $S \times S \times \ldots \times S$. The $(\ )_n$-subsets, and the equivalence decompositions of the subset of the Cartesian product, may be considered to be induced by, and are characterized in terms of, a linear space D(S) of real-valued functions on the set S, or an arbitrary linear subspace M of such a space.

A later publication will be concerned with applications of the results of this paper. For example, when the space D(S) is the space C(S) of all real continuous functions on a topological space S, the partitions possibly induced by linear subspaces $M \subset C(S)$ reveal topological characteristics of S, and in particular for finite dimensional M, properties of mappings on S to Euclidean spaces. (See [1].)

2. *An Equivalence Relation; One-Algebraic Homogeneity*

Denote by D(S) an arbitrary linear space of real functions on the set S. For any linear subspace N of D(S), call a subset $\{s_1, \ldots, s_n\}$ a *set of rank* n for N (or an n-*rank set*), and say that the n points are n-*separated* by N, if arbitrary real values $r_1, \ldots, r_n$ may be fitted at $s_1, \ldots, s_n$ by some function from N; i.e., if given any $r_1, \ldots, r_n$, there always is a function $x \varepsilon N$ such that $x(s_1) = r_1, \ldots, x(s_n) = r_n$. Define $(s, s')_2$ in case $\{s, s'\}$ is not a set of rank 2 for N; $(s, s', s'')_3$ in case $\{s, s', s''\}$ is not a set of rank 3 for N; .... Let us say that a subspace N of D(S) is n-*algebraically homogeneous* in case each subset $\{s_1, \ldots, s_n\}$ of n different points of S is a set of rank n for N. Clearly if N is n-algebraically homogeneous, it must be at least n-dimensional. If N is n-algebraically homogeneous, then it is also k-algebraically homogeneous for each $k < n$.

Let $O = O(N)$ be the subset of S on which all functions of N vanish, and denote the empty set by φ. Then for a linear subspace N, the condition $O = \varphi$ is satisfied if and only if N is 1-algebraically homogeneous.

THEOREM 1. *In case* $O = \varphi$, *the relation* $(\ )_2$ *is an equivalence relation in* S.

Proof. Suppose $(s, s')_2$ and $(s', s'')_2$. Then there exist relations of linear dependence $ax(s) + a'x(s') = 0$, $b'x(s') + b''x(s'') = 0$, for all $x \varepsilon N$. Eliminating $x(s')$, we obtain $b'ax(s) - a'b''x(s'') = 0$. This is a relation of linear dependence, so that $(\ )_2$ is transitive, unless both $b'a = 0$ and $b''a' = 0$. But the latter imply either $b' = 0$ and $a' = 0$, or $a = 0$ and $b'' = 0$, since b', b'' are not both zero, and a, a' are not both zero. If $b' = 0$, $a' = 0$, then $x(s)$ and $x(s'')$ are identically zero for all $x \varepsilon N$, contrary to the hypothesis that $O = \varphi$. Similarly, if $a = 0$, $b'' = 0$, then $x(s') = 0$ for all $x \varepsilon N$, again contradicting the hypothesis. Therefore $(\ )_2$ is an equivalence relation.

In case N contains a nowhere-vanishing function, then obviously we have $O = \varphi$. For an infinite set S, if N is the subspace of all finite linear combinations of the characteristic functions of the points of S, then $O = \varphi$, but N does not contain a nowhere-vanishing function.

LEMMA 1. *If two functions* x, y *on a set* S *have respective cozero-sets (complements of sets on which they vanish)* A, B, *with* $S = A \cup B$, *and if the cardinality of* $A \cap B$ *is less than the power of the continuum, then there exists a linear combination* $ax + by$ *which vanishes nowhere on* S.

Proof. Any linear combination $ax + by$ with $a \neq 0$ and $b \neq 0$ does not vanish on $(A - B) \cup (B - A)$. For fixed $a \neq 0$, the equation $ax(s) + b'y(s) = 0$ has a solution $b' = b'(s) \neq 0$, for each $s \varepsilon A \cap B$. By hypothesis, the set of solutions $\{b'\}$ cannot be the set of all non-zero real numbers; therefore there exists a $b \neq 0$ such that $ax + by$ does not vanish on $A \cap B$. Since both a and b are different from zero, the function $ax + by$ vanishes nowhere on S, as required. Even if $A \cap B$ has the power of the continuum (or higher power), a linear combination $ax + by$ which vanishes nowhere on S exists unless $x(s)/y(s)$, like the real or imaginary part of a complex analytic function in the neighborhood of an essential singularity, assumes every possible non-zero value on $A \cap B$.

THEOREM 2. *If a finite set of functions* $x_1, \ldots, x_m$ *from a subspace* N *have respective cozero-sets* $A_1, \ldots, A_m$, *which form a partition of* S, *such that for each* i, j, $i \neq j$, *the intersection* $A_i \cap A_j$ *has cardinality less than the power of the continuum, then there exists a function* $y \varepsilon N$ (*a linear combination of* $x_1, \ldots, x_m$) *which vanishes nowhere on* S.

Proof. By hypothesis and by Lemma 1, there exists a linear combination $a_1x_1 + a_2x_2$ which does not vanish on $(A_1 \cup A_2)$. Then again by hypothesis and by Lemma 1, there exists a linear combination of $(a_1x_1 + a_2x_2)$ and of x_3 which vanishes nowhere on $(A_1 \cup A_2) \cup A_3$; continuing in this way, we obtain the required y as a linear combination of $x_1, \ldots, x_m$.

COROLLARY 1. *In the space* C(S) *of all real continuous functions on a topological space* S, *let* N *be any linear subspace which does not contain a unit (any ideal of the ring* C(S), *for example) and which is such that there is a covering of* S *by cozero-sets of functions from* N. *Then in case there is a covering of* S *by a finite subset* $A_1, \ldots, A_m$ *of cozero-sets, there exists a pair* A_i, A_j, $i \neq j$, *whose intersection* $A_i \cap A_j$ *has at least the power of the continuum. The corresponding functions* x_i, x_j *are such that* $x_i(s)/x_j(s)$ *assumes every non-zero real value on* $A_i \cap A_j$.

In case S is compact, of course finite subcoverings do exist (but since each ideal in C(S) then is fixed [1], any subspace N as described in the Corollary is not contained in an ideal, and therefore generates the entire ring C(S)).

3. *Separation of Points*

If a subspace N of D(S) contains constant functions, then $O = \varphi$, and the relation $(\ )_2$

for N is the ordinary relation of nonseparation of points of S. That is, in case N contains constants and separates points of S, then the $(\ )_2$-subsets are singletons.

THEOREM 3. *For any 2-algebraically homogeneous space* N, *the* $(\ )_2$*-subsets are singletons. If a subspace* N *contains the constant functions and separates points, then it is 2-algebraically homogeneous. A subspace* N *may separate the points of* S, *without being 2-algebraically homogeneous, or even 1-algebraically homogeneous. If a subspace* N *is 2-algebraically homogeneous, then it separates points, but it does not need to contain the constant functions.*

Proof. If N is 2-algebraically homogeneous, then $O = \varphi$ since a 2-algebraically homogeneous space must be 1-algebraically homogeneous. By Theorem 1, $(\ )_2$ is an equivalence relation, and by the hypothesis that N is 2-algebraically homogeneous, $(s, s')_2$ if and only if $s = s'$. If N separates points and contains constants, then if $s \neq s'$, $x(s) \neq x(s')$, some scalar multiple ax will realize a prescribed difference of values at s, s′, and for a suitable constant c, c + ax will fit the prescribed values at s, s′. Therefore N is 2-algebraically homogeneous. A function which has different values at all points of S separates the points of S, so if the cardinality of S does not exceed the power of the continuum, the points of S may be separated by a one-dimensional subspace N. A single separating function may assume the value zero at one point. For S of larger cardinality, there may of course be a decomposition of S into sets of cardinal numbers not exceeding the power of the continuum, and corresponding functions in N which separate points, such that no set of two points from the same subset of the decomposition are 2-separated by N. In case N is 2-algebraically homogeneous, then for s, s′, $s \neq s'$, of course there is an $x \in N$ with $x(s) \neq x(s')$. The subspace N of all finite linear combinations of characteristic functions of the points of an infinite set S is n-algebraically homogeneous for each n, but does not contain constants.

Let S be the circumference $0 \leq s \leq 2\pi$ where $s = 0$ and $s = 2\pi$ are identified. The two-dimensional subspace N, spanned by the functions sin s, cos s on S, separates points of S, but does not contain constants, and is not 2-algebraically homogeneous. The $(\ )_2$-subsets consist of pairs of antipodal points (points s, s′ such that $s - s' = \pm \pi$).

4. *Subsets of Mutually* $(\ )_n$*-Related Elements*

Recall the definitions of n-rank set and of $(\ )_n$-subsets at the beginning of Section 2.

THEOREM 4. *Suppose* that $O = \varphi$. *If* $(s, s', s'')_3$ *and* $(s, t)_2$, $(s', t')_2$, $(s'', t'')_2$, *then* $(t, t', t'')_3$. *Thus the relation* $(\ )_3$ *may be considered to be defined for the* $(\ )_2$*-subsets as elements, with* s, s′, s″ *replaced by the respective* $(\ )_2$*-subsets which they represent.*

Proof. Since by hypothesis $O = \varphi$, neither of s, t is $(\ )_1$. Therefore there is a relation of linear dependence $ax(s) + bx(t) = 0$, with $a \neq 0$, $b \neq 0$. Since $(s, s', s'')_3$, we have $cx(s) + c'x(s') + c''x(s'') = 0$, with c, c′, c″ not all zero. Replacing x(s) by $(-b/a)x(t)$, we obtain $(t, s', s'')_3$. Similarly s′ may be replaced by t′, and s″ by t″.

THEOREM 5. *Suppose that* $O = \varphi$. *If* $s_2, \ldots, s_n$ *are not* $(\ )_{n-1}$*-related (i.e., if* $\{s_2, \ldots, s_n\}$ *is an* $(n - 1)$*-rank set), and if* $(s_1, \ldots, s_n)_n$, $(s_2, \ldots, s_{n+1})_n$, *then* $s_1, \ldots, s_{n+1}$ *are mutually* $(\ )_n$*-related (i.e., each subset of* n *of* $s_1, \ldots, s_{n+1}$ is $(\ )_n$*-related).*

Proof. By hypothesis, there are relations of linear dependence $a_1x(s_1) + \ldots + a_nx(s_n) = 0$, $b_2x(s_2) + \ldots + b_{n+1}x(s_{n+1}) = 0$, in which $a_1 \neq 0$. For if a_1 (or b_{n+1}) were zero, $\{s_2, \ldots, s_n\}$ would not be an $(n - 1)$-rank set. By combining these equations, $x(s_2), \ldots, x(s_n)$ may be eliminated in turn, to obtain a relation of linear dependence, since $a_1 \neq 0$. This implies $(s_1, s_3, \ldots, s_{n+1})_n, \ldots, (s_1, \ldots, s_{n-1}, s_{n+1})_n$. (If a coefficient of $x(s_i)$ in one of the equations is zero, that equation already implies that the other n s_j's are $(\ )_n$-related.)

COROLLARY 2. *If* $A = B \cup \{s_2, \ldots, s_n\}$, *where* $\{s_2, \ldots, s_n\}$ *is an* $(n - 1)$*-rank set, and if each* $s \in B$ *is such that* $(s, s_2, \ldots, s_n)_n$, *then A is a mutually* $(\ )_n$*-related subset.*

Proof. Since $\{s_2, \ldots, s_n\}$ is an $(n - 1)$-rank set, for $t_1, \ldots, t_k \in B$, by hypothesis we have relations of linear dependence $x(t_1) + a_2x(s_2) + \ldots + a_nx(s_n) = 0, \ldots, x(t_k) + d_2x(s_2) + \ldots + d_nx(s_n) = 0$. Eliminate $x(s_2)$ from $(k - 1)$ pairs of these equations; then $x(s_3)$ from $(k - 2)$ of the resulting equations; and so on to obtain a relation of linear dependence involving $x(t_1), \ldots, x(t_k)$, $k \leq n$, which shows that $(t_1, \ldots, t_k, \ldots, s_n)_n$. Similarly any $(k - 1)$ of the $x(s_i)$'s may be eliminated.

THEOREM 6. *Any proper maximal subset of mutually* $(\ )_n$*-related elements contains an* $(n - 1)$*-rank set.*

Proof. For suppose that s is outside the maximal subset. Then there exist some $(n - 1)$

elements in the subset which are not $(\ )_n$-related with s. This implies that s and the $(n-1)$ elements are an n-rank set. Therefore in particular the $(n-1)$ elements are an $(n-1)$-rank set.

5. *Equivalence of Rank Sets*

In this section we introduce an equivalence relation in the subset of n-rank sets of the set $S \times S \times \ldots \times S$ (Cartesian product of S with itself n times). For a subspace M of D(S), consider the dual space M^* of all real linear functionals on M. For each fixed $s \varepsilon S$, $s^*(x) = x(s)$ is an element of M^*. Define a mapping φ on S to M^* by $\varphi(s) = s^*$. Clearly the mapping φ is one-to-one if and only if M separates the points of S, and M is 1-algebraically homogeneous if and only if no point of S is mapped by φ into the origin of M^*.

Call the singletons {s}, where the point s is such that $(s)_1$ (i.e., such that all functions $x \varepsilon M$ vanish at s), 0-rank sets. If {s} is not a 0-rank set, it is a 1-rank set. If M separates points, then {s} is a 0-rank set for at most one point s of S. If M is one-algebraically homogeneous, then there is no s such that {s} is a 0-rank set. Suppose that M is 1-algebraically homogeneous, but does not necessarily separate points of S. Two 1-rank sets $\{s_1\}$, $\{s_2\}$ are such that $x(s_2) = cx(s_1)$, c fixed, for all $x \varepsilon M$, if and only if $(s_1, s_2)_2$.

DEFINITION. Two n-rank sets, $\{s_1, \ldots, s_n\}$ and $\{t_1, \ldots, t_n\}$, are $\sim$ *related* if and only if $s_1^*, \ldots, s_n^*$, and $t_1^*, \ldots, t_n^*$, span the same linear subspace of M^*. Relation $\sim$ evidently is an equivalence relation on the subset of n-rank sets of $S \times S \times \ldots \times S$. The two n-rank sets are λ-related in case there is a common n-dimensional subspace E_n of M which fits arbitrary values at $s_1, \ldots, s_n$ and also at $t_1, \ldots, t_n$. (The reflexive and symmetric relation λ is not an equivalence relation.)

THEOREM 7. *For an n-rank set* $\{s_1, \ldots, s_n\}$, *the linear functionals* $s_1^*, \ldots, s_n^*$ *span an n-dimensional linear subspace* E_n^* *of* M^*.

Proof. By the hypothesis that $\{s_1, \ldots, s_n\}$ is an n-rank set, there exist functions $x^{(1)}, \ldots, x^{(n)} \varepsilon M$ such that the determinant $|\{x^{(i)}(s_j)\}|$ does not vanish. Thus $|\{s_j^*(x^{(i)})\}| \neq 0$, which implies that $\{x^{(1)}, \ldots, x^{(n)}\}$ is an n-rank set for the subspace E_n^* of M^* which is spanned by $s_1^*, \ldots, s_n^*$. Therefore E_n^* is n-dimensional.

THEOREM 8. *The equivalence relation $\sim$ refines (i.e., implies) the relation λ. That is, whenever for two n-rank sets $\{s_j\}$, $\{t_j\}$, we have $\{s_j\} \sim \{t_j\}$, it follows that $\{s_j\}\ \lambda\ \{t_j\}$.*

Proof. Let E_n^* be the n-dimensional subspace of M^* which is spanned by $s_1^*, \ldots, s_n^*$ and also by $t_1^*, \ldots, t_n^*$. By Theorem 1 of [2], there exist n functions $x^{(1)} \ldots, x^{(n)} \varepsilon M$ which form an n-rank set for E_n^*. Accordingly the determinants $|\{s_i^*(x^{(j)})\}| = |\{x^{(j)}(s_i)\}|$ and $|\{t_i^*(x^{(j)})\}| = |\{x^{(j)}(t_i)\}|$ do not vanish; that is, the n-rank sets $\{s_j\}$ and $\{t_j\}$ both are n-separated by the n-dimensional subspace E_n of M which is spanned by the functions $x^{(1)}, \ldots, x^{(n)}$.

THEOREM 9. *Suppose that $O = \varphi$ and M is finite dimensional. If $\{s_1, \ldots, s_k\}$ is a k-rank set, representative of a $(\ )_{k+1}$-subset, then all the functions x of the subspace $L = \{x \varepsilon M: x(s_j) = 0, j = 1, \ldots, k\}$ vanish at a point $t \varepsilon S$ if and only if t belongs to the $(\ )_{k+1}$-subset. Thus a $(\ )_{k+1}$-subset is uniquely determined by each k-rank set which it contains, and $\{s_1, \ldots, s_k\}$ is a singleton k-rank set if and only if for each $t \varepsilon S\text{-}\{s_1, \ldots, s_k\}$, there exists a function x of L such that $x(t) \neq 0$.*

Proof. Since M is finite dimensional, by Theorem 1.9–G on p. 51 of [4], the subspace K^* which is spanned by $s_1^*, \ldots, s_k^*$ is algebraically saturated [4]. Therefore it follows necessarily, by Theorem 1.9–E on p. 50 of [4], that if $x(t) = 0$ for all $x \varepsilon L$, then $t^* \varepsilon K^*$, and conversely that if $t^* \varepsilon K^*$, then $x(t) = 0$ for all $x \varepsilon L$.

THEOREM 10. *Suppose that $O = \varphi$. Each proper maximal $(\ )_{k+1}$-subset consists of those elements of S which are contained in the k-rank sets of a $\sim$ equivalence subset of k-rank sets.*

Proof. By Theorem 6, the $(\ )_{k+1}$-subset contains a k-rank set $\{s_1, \ldots, s_k\}$. If $t \notin \{s_1, \ldots, s_k\}$ but t is in the $(\ )_{k+1}$-subset, then $(s_1, \ldots, s_k, t)_{k+1}$, so that $x(t) = c_1x(s_1) + \ldots + c_kx(s_k)$ for all $x \varepsilon M$; i.e., $t^* = c_1s_1^* + \ldots + c_ks_k^*$. Therefore t^* is contained in the linear span K^* of $s_1^*, \ldots, s_k^*$, and t^* together with some $(k-1)$ of $s_1^*, \ldots, s_k^*$ span K^*. If $x^{(1)}, \ldots, x^{(k)} \varepsilon M$ are a k-rank set for K^*, then the determinant, which has for columns $\{x^{(i)}(t)\}$ and $(k-1)$ of the k columns $\{x^{(i)}(s_j)\}$, does not vanish. Therefore t and the $k-1$ of $s_1, \ldots, s_k$ are a k-rank set. In case a k-rank set $\{t_1, \ldots, t_k\}$ is equivalent to $\{s_1, \ldots, s_k\}$, then since K^* is k-dimensional, we have $(s_1, \ldots, s_k, t_j)_{k+1}$ for $j = 1, \ldots, k$. The theorem is proved.

In case K^* is not algebraically saturated, then there may exist a t, not in the $(\ )_{k+1}$-subset which is determined by the k-rank set $\{s_1, \ldots, s_k\}$, such that t^* vanishes at all functions where $s_1^*, \ldots, s_k^*$ do, but t^* is not contained in K^*. The characterization of Theorem

9, of a $(\ )_{k+1}$-subset as the subset of elements of S, where the functions x of M which vanish at a k-rank subset $\{s_1, \ldots, s_k\}$ of the $(\ )_{k+1}$-subset, must also vanish, seems to be available only when the subspace M is finite dimensional.

DEFINITION. Two k-rank sets, $\{s_1, \ldots, s_k\}$, $\{t_1, \ldots, t_k\}$, are $\approx$ *related* in case the subspaces L of functions which vanish at $s_1, \ldots, s_k$, and L_1 of functions which vanish at $t_1, \ldots, t_k$, coincide.

THEOREM 11. *The equivalence relation* $\sim$ *is a refinement of the equivalence relation* $\approx$, *which in turn is a refinement of the relation* λ. (See Theorem 8.) *In case* M *is finite dimensional, the two equivalence relations coincide.*

Proof. The first and last statements are evident after the paragraph preceding the definition of $\approx$, in consequence of Theorem 9. If t^* is a linear combination of $s_1^*, \ldots, s_k^*$, then $x(t)$ is a linear combination with the same coefficients of $x(s_1), \ldots, x(s_k)$, so $x(t)$ vanishes for all $x \in L$. If $\{s_1, \ldots, s_k\} \approx \{t_1, \ldots, t_k\}$, then the k-rank sets $\{s_1, \ldots, s_k\}$, $\{t_1, \ldots, t_k\}$ not only have a common fitting subspace E_k of M, but also the functions of E_k *simultaneously* fit arbitrary values of functions of M at $s_1, \ldots, s_k$ and at $t_1, \ldots, t_k$. Any algebraic complement L in M of E_k thus must consist of all functions which vanish at $s_1, \ldots, s_k$ and also at $t_1, \ldots, t_k$. (See [3].)

Call a subspace E_k M a *fitting space* for a subset R of S, in case for each function z of M, there is a function x of E_k which coincides with z on R. In case a pair of k-rank sets $\{s_1, \ldots, s_k\}$ and $\{t_1, \ldots, t_k\}$ are λ-related but not $\approx$ related, then they have a common fitting space E_k which is not a simultaneous fitting space, i.e., not a fitting space for the subset $\{s_1, \ldots, s_k, t_1, \ldots, t_k\}$. (To fit arbitrary values at $s_1, \ldots s_k, t_1, \ldots, t_k$, a subspace of M would have to be at least (2k)-dimensional.) The relation of having a simultaneous fitting space is the same as $\approx$.

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