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NUMBERS**

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Reprinted from the  
**BULLETIN OF THE AMERICAN MATHEMATICAL SOCIETY**  
May, 1966, Vol. 72, No. 3  
pp. 512-514

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## SOME UNSYMMETRIC COMBINATORIAL NUMBERS

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Communicated by V. Klee, January 26, 1966

By an *n*-configuration we shall mean an abstract set of *n* elements, together with the set of all unordered pairs of distinct elements from the set. It is convenient also to use quasi-geometrical terminology such as *vertex* for element, *edge* or *side* for a pair (2-tuple), *triangle* as well as triple (3-tuple) for a 3-subconfiguration, and so on.

The Ramsey number  $N(p, q, 2)$  (see [3, pp. 38-43], or [2, pp. 61-65]), for two kinds *h*, *v* of pairs (or two "colors of edges"), is the smallest integer such that if  $n \geq N(p, q, 2)$ , then any *n*-configuration is sure to contain *either* an *h* *p*-tuple (a *p*-tuple all of whose edges are *h*) or a *v* *q*-tuple. Call a *p*-tuple all of whose edges are alike (*h* or *v*) a *like p*-tuple. We introduce, and partially determine the values of, new analogous combinatorial numbers  $K(p, q, 2)$ ,  $M(p, q, 2)$ , and  $V(p, q, 2)$ .

DEFINITIONS. The number  $K(p, q, 2)$  is the smallest integer such that if  $n \geq K(p, q, 2)$ , then for each vertex, the configuration is sure to contain *either* a like *p*-tuple containing the vertex, or a like *q*-tuple not containing the vertex. For three kinds *r*, *g*, *v* of edges,  $M(p, q, 2)$

is the smallest integer such that if  $n \geq M(p, q, 2)$ , the configuration is sure to contain either a like  $p$ -tuple, or a  $j, k$   $q$ -tuple (a  $q$ -tuple having at most two kinds  $j, k$  of edges, where  $j, k = r, g, \text{ or } v$ ). The number  $V(p, q, 2)$  is the smallest integer such that if  $n \geq V(p, q, 2)$ , then for each vertex of the configuration, the configuration contains either a like  $p$ -tuple containing the vertex, or a  $j, k$   $q$ -tuple not containing the vertex.

Consider for a moment "vertical" numbers, which otherwise are like the Ramsey numbers:  $S(p, q, 2)$ , for example, is the smallest integer such that a configuration with  $n \geq S(p, q, 2)$  is sure to contain, for each vertex, either an  $h$   $p$ -tuple containing the vertex, or a  $v$   $q$ -tuple not containing the vertex. Evidently  $N(p, q, 2) \leq S(p, q, 2)$ . But for all  $p \geq 3, q \geq 3, S(p, q, 2) = \infty$ : for arbitrarily large  $n$ , at one vertex, assign  $(p-1)$  edges from the vertex to be  $h$ , the remainder  $v$ . Let one edge joining a pair of other ends of the  $(p-1)$  edges be  $v$ , and let all other edges of the  $n$ -configuration be  $h$ . Then for the vertex, the  $n$ -configuration contains neither an  $h$   $p$ -tuple containing the vertex, nor a  $v$   $q$ -tuple not containing the vertex. Moreover  $S(p, q, 2) = \infty$  for all  $p \geq 2, q \geq 2$ .

Denote by  $W(q, p, 2)$  the smallest integer such that if  $n \geq W(q, p, 2)$ , then for each vertex the configuration is sure to contain either a  $j, k$   $q$ -tuple containing the vertex, or a like  $p$ -tuple not containing the vertex. We notice that  $W(q, p, 2) = V(p, q, 2)$ .

Our results so far concerning the numbers  $K, M, V$  are indicated in the following Theorem 1 (including the table) and Theorem 2. For purposes of comparison, the known values of the Ramsey numbers  $N$  also are included: the entries in the table are the values of  $N, K, M, V$  in that order, for each  $p, q$ .

**THEOREM 1.** For  $p, q$  from 3 to 5 inclusive, the numbers have values as given in the following table (cf. the table in [3, p. 42]).

$p \backslash q$	3	4	5
3	6, 6, 5, 6	9, 8, 8, 10 or 11	14, 10, 14, 14
4	9, 7, 5, 6	18, 18, 10 to 17, 11 to 18	
5	14, 7, 5, 6		

For all  $p > 3$ , we have that  $K(p, 3, 2) = 7$ ; for all  $q \geq 3$ , that  $K(q, q, 2) = N(q, q, 2)$  and  $K(3, q, 2) = 2q$ ; for  $p > q$ , that  $K(p, q, 2) \leq N(q, q, 2) + 1$ ; and for  $q > p > 3$ , that  $K(p, q, 2) \geq \max(K(p-1, q, 2) + 1, 2q + p - 3)$ . Further,  $N(3, 6, 2)$  is 17 or 18 (cf. [1]), and  $17 = M(3, 6, 2) \leq N(3, 6, 2)$ .

A configuration is called *degenerate, with respect to* any of the combinatorial numbers, in case it does contain (for each vertex in case of  $V$ ) either a  $p$ -tuple or a  $q$ -tuple as described in the corresponding definition. As an example, it is quite easy to find an 8-configuration which is nondegenerate with respect to  $N(3, 4, 2)$ —an octagon with an 8-cycle plus a 4-cross (see below) of blue edges, and an 8-cycle plus two 4-cycles of red edges, has neither a blue triangle nor a red quadruple (cf. the existence proof for the 8-configuration in [1]). The method of establishing the above lower bounds  $L$  is to exhibit in each case a nondegenerate configuration with  $n = L - 1$ . To establish an upper bound  $U$ , it is sufficient to show that any configuration with  $n = U$  must be degenerate. The value of a combinatorial number of course is determined in case  $L = U$ . Details will be included in a paper which will be offered for publication elsewhere.

A subsidiary result, analogous to Steiner triple systems ([2] or [3]), is the following. A  $k$ -cycle is a closed string of  $k$  successively adjacent edges, such as  $12; 23; \dots; k-1, k; k, 1$ ; where  $\{1, \dots, k\}$  is a subset of  $k$  of the vertices of the configuration. In any  $(2n+1)$ -configuration, the edges can be covered (each exactly once) by  $n$   $k$ -cycles with  $k = (2n+1)$ . A  $k$ -cross is a set of  $k$  edges, no two of which are adjacent. The edges of any  $(2n+2)$ -configuration can be covered by  $n$   $(2n+2)$ -cycles and an  $(n+1)$ -cross.

**THEOREM 2.** *We have  $K(p, q, 2) \leq N(p, q, 2)$ ,  $M(p, q, 2) \leq N(p, q, 2)$ ,  $M(p, q, 2) \leq V(p, q, 2)$ . For each  $q$ ,  $V(q, q, 2)$  is either  $M(q, q, 2)$  or  $M(q, q, 2) + 1$ ; for any  $p$ ,  $V(p, q, 2) \leq M(q, q, 2) + 1$ . For  $p > q$ ,  $M(q, q, 2) \leq \min(M(p, q, 2), V(p, q, 2))$ , and  $V(q, q, 2) \leq V(p, q, 2)$ . For  $p \geq 3$ ,  $M(p, 3, 2) = 5$ , and  $V(p, 3, 2) = 6$ . For  $q > 3$ ,  $V(3, q, 2) \leq (3q - 1)$ .*

Reference [4], for example, indicates the wealth of possible applications for combinatorial results.

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