Null Vector Number Systems

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Abstract

Let $\mathcal{U}_\mathcal{F} := gen\{\mathcal{A}\cup\mathcal{B}\cup\ldots\}_\mathcal{F}$ be the *universal algebra* generated by taking the addition and multiplication of null vectors in the union of sets of anti-commuting null vectors: $\mathcal{U}^1_\mathcal{F} := \cup\{\mathcal{A},\mathcal{B},\ldots\}_\mathcal{F}$. Since the null vectors in each of these sets are anti-commuting, the products can be taken to be the anti-symmetric Grassmann *wedge product.* It follows that each such set generates a corresponding Grassmann algebra $\mathcal{G}(\mathcal{A}),\mathcal{G}(\mathcal{B}),\ldots$ over same field $\mathcal{F}$. These null vectors, and their doubles, are used to define basic double number fields of real numbers, complex numbers, and quaternions, used in the construction of all real and complex Clifford geometric algebras of pseudo-riemannian vector spaces. The null vector classification of the most important lower dimensional geometric algebras, and their isomorphic coordinated matrix algebras, are discussed in detail independent of the concept of a metric. The structure of all higher dimensional geometric algebras can then be extrapolated.

1 Null Vector Lines and Rays

A real null vector represents an *oriented direction* of a *ray* of light. Since it travels at the speed of light in the 3-D Euclidean *rest-frame* of every observer, a null-vector is best visualized by an oriented line having *direction* but *undefined* length. Welcome to the strange world of Minkowski spacetime in which the concept of an *oriented direction* is independent from the concept of a definite metric.

Let $\mathbf{c} \in \mathcal{U}^1_\mathcal{F} \subset \mathcal{U}_\mathcal{F}$ be a null vector in the generating set $\mathcal{U}^1_\mathcal{F}$ of all such null vectors, with the defining property that $\mathbf{c}^2 = 0$. The *universal algebra* $\mathcal{U}_\mathcal{F}$, over a field $\mathcal{F}$, is generated by taking the *addition* and *multiplication* of null vectors in $\mathcal{U}^1_\mathcal{F}$, according to the familiar rules of addition and multiplication of matrices over a field $\mathcal{F}$. It follows that

$$\mathcal{U}_\mathcal{F} := gen\{\mathcal{U}^1_\mathcal{F}\} = gen\{\mathcal{A}\cup\mathcal{B}\cup\mathcal{C}\cup\ldots\}_\mathcal{F}$$

where $\mathcal{U}^1_\mathcal{F} := \{\mathcal{A}\cup\mathcal{B}\cup\mathcal{C}\cup\ldots\}_\mathcal{F}$ is the union of all such sets of anti-commuting null vectors. Since the null vectors in each these sets are anti-commuting, multiplication of null vectors reduces to the *wedge products* in the Grassmann algebras

$\mathcal{G}(\mathcal{A}), \mathcal{G}(\mathcal{B}), \ldots$, respectively. Sets of the type $\mathcal{C}_k := \cup\{\{\mathbf{c}_1\}, \ldots, \{\mathbf{c}_k\}\}$ consisting of singleton null vectors, generate k distinct 2^1-dimensional Grassmann algebras over $\mathcal{F}$.

Nullvector Number Line

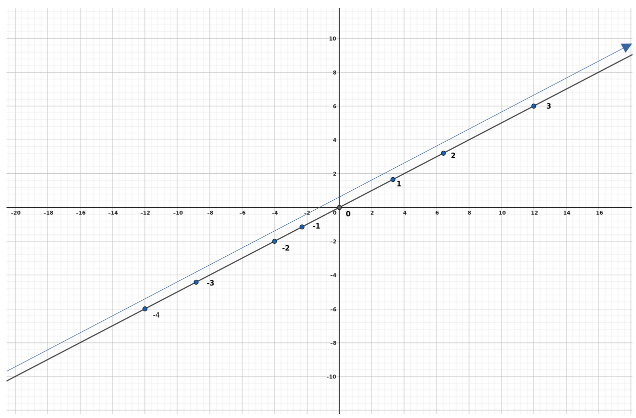

Figure 1: The concept of an oriented ray is independent of the concept of a definite metric. A *scale* which respects the natural ordering of the real numbers is shown. Also shown is the double ray $^2\mathbf{c}$. The second ray (blue) has an oriented direction but an undetermined scale.

Since $\mathbf{c}$ is a null vector, or ray, by introducing a *scale* it becomes an *oriented real null vector line* $\mathbb{R}_\mathbf{c}$. First, pick out a zero point 0 called the *origin*. Then choose points along the line naming the positive integers $1, 2, 3, 4, ...$, moving in the positive direction along the ray, and the negative integers $-1, -2, -3, ...$ naming the points in the negative direction. The points need not be evenly spaced because for any real number $r \in \mathbb{R}$, $(r\mathbf{c})^2 = r^2\mathbf{c}^2 = 0$. Once a scale has been chosen, the construction of $\mathbb{R}_{\mathbf{c}}$ is complete. Taken together over the real numbers, gives the singleton Grassmann algebra

$$\mathcal{G}_{\mathbb{R}}(\mathbf{c}) := \mathbb{R} \oplus \mathbb{R}_{\mathbf{c}} \subset \mathcal{U}_{\mathcal{F}}. \tag{1}$$

The Grassmann algebra $\mathcal{G}_{\mathbb{R}}(\mathbf{c})$ is equivalent to the geometric algebra $\mathbb{G}_{0,0,1}(\mathbb{R})$, and for real numbers $r, s, t \in \mathbb{R}$, $r\mathbf{c} = \mathbf{c}r$,

$$r\mathbf{c} + s\mathbf{c} = (r+s)\mathbf{c}, \quad r(s\mathbf{c}) = (rs)\mathbf{c}, \quad t(r\mathbf{c} + s\mathbf{c}) = (tr)\mathbf{c} + (ts)\mathbf{c}. \tag{2}$$

1.1 Double null vectors and rays

The concept of a *double null vector*, or *double ray*, over the real number field $\mathbb{R}$ is important.

$$^2\mathbf{c}_{\mathbb{R}} := \{(r\mathbf{c}, s\mathbf{c}) |\ r, s \in \mathbb{R}\}. \tag{3}$$

A real double ray $^2\mathbf{c} \equiv (r,s)\mathbf{c}$ shares the same direction and origin 0, but can have a different orientation and scale. Double rays are added and multiplied component-wise. For $^2\mathbf{c}_1 = (r_1, s_1)\mathbf{c}$ and $^2\mathbf{c}_2 = (r_2, s_2)\mathbf{c}$

$$^2\mathbf{c}_1 + {}^2\mathbf{c}_2 = (r_1 + r_2, s_1 + s_2)\mathbf{c},\ {}^2\mathbf{c}_1\, {}^2\mathbf{c}_2 = (r_1 r_2, s_1 s_2)\mathbf{c}. \tag{4}$$

Figure 1 pictures the real *double ray* $^2\mathbf{c}$ as a pair of parallel lines through the origin 0. They are drawn as separate lines close together, although they are the same line distinguished only by possibly different scales.

A similar construction can be given for the complex null vector line $\mathbb{C}_j(\mathbf{c})$, its corresponding complex Grassmann algebra, and its complex geometric algebra,

$$\mathcal{G}_{\mathbb{C}_j}(\mathbf{c}) := \mathbb{C}_j \oplus \mathbb{C}_j(\mathbf{c}) \equiv \mathbb{G}_{0,0,1}(\mathbb{C}_j) \subset \mathcal{U}_{\mathbb{C}_j}. \tag{5}$$

In this case *scalers* are the *complex numbers*

$$\mathbb{C}_j := \{x + jy |\ x, y \in \mathbb{R}, j := \sqrt{-1}\}.$$

The complex scalers $\mathbb{C}_j$ of a complex ray $\mathbf{c} := \mathbf{c}(\mathbb{C}_j)$ represent two *intrinsic* degrees of freedom. Just like a real double null vector $^2\mathbf{c}_{\mathbb{R}}$ in (4),

$$^2\mathbf{c}_{\mathbb{C}} := \{(z\mathbf{c}, w\mathbf{c}) |\ z, w \in \mathbb{C}_j\}, \tag{6}$$

and are added and multiplied component-wise. The the complex double ray $^2\mathbf{c} = (z, w)\mathbf{c}$ share the same direction and origin 0, but can have a different scales just as in the case of a real double ray pictured in Figure 1.

1.2 Simple geometric algebras of rays

Let $\mathbb{G}_{0,0,1}(\mathbf{c}_1) := span\{1, \mathbf{c}_1\}_{\mathbb{R}}$, $\mathbb{G}_{0,0,1}(\mathbf{c}_2) := span\{1, \mathbf{c}_2\}_{\mathbb{R}}$ and $\mathbb{G}_{0,0,1}(\mathbf{c}_3) := span\{1, \mathbf{c}_3\}_{\mathbb{R}}$ be three geometric algebras of the rays $\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3 \in \mathcal{U}_{\mathbb{R}}$, as defined in (1). Suppose that the corresponding singleton Grassmann algebras are *correlated* by the rule

$$\mathbf{c}_i \cdot \mathbf{c}_j := \frac{1}{2}(\mathbf{c}_i\mathbf{c}_j + \mathbf{c}_j\mathbf{c}_i) = \frac{1}{2}(1 - \delta_{ij}). \tag{7}$$

Easy consequences follow from the rule (7). Defining $\mathbf{e} := \mathbf{c}_1 + \mathbf{c}_2$ and $\mathbf{f} = (\mathbf{c}_1 - \mathbf{c}_2)$,

- $\mathbf{e}^2 = (\mathbf{c}_1\mathbf{c}_2 + \mathbf{c}_2\mathbf{c}_1) = 1$, $\mathbf{f}^2 = -1$ and $\mathbf{ef} = -\mathbf{fe}$.

  The orthonormal quantities $\mathbf{e}$ and $\mathbf{f}$, constructed from the null vectors $\mathbf{c}_1$ and $\mathbf{c}_2$, are called *Euclidean* and *pseudo-Euclidean* unit vectors, respectively. In addition, the new quantity

- $\mathbf{ef} = (\mathbf{c}_1 + \mathbf{c}_2)(\mathbf{c}_1 - \mathbf{c}_2) = \mathbf{c}_2\mathbf{c}_1 - \mathbf{c}_1\mathbf{c}_2 = 2\mathbf{c}_2 \wedge \mathbf{c}_1$ is a unit *hyperbolic bivector*, since $(\mathbf{ef})^2 = 4(\mathbf{c}_2 \wedge \mathbf{c}_1)^2 = (\mathbf{c}_2\mathbf{c}_1 - \mathbf{c}_1\mathbf{c}_2)^2 = 1$.

- $\mathbf{c}_1\mathbf{c}_2\mathbf{c}_1 = \mathbf{c}_1$, $\mathbf{c}_2\mathbf{c}_1\mathbf{c}_2 = \mathbf{c}_2$, and $(\mathbf{c}_1\mathbf{c}_2)^2 = \mathbf{c}_1\mathbf{c}_2$.

- The geometric algebra

$$\mathbb{G}_{1,1} := span\{1, \mathbf{e}, \mathbf{f}, \mathbf{ef}\}_{\mathbb{R}}. \tag{8}$$

- Alternatively, the geometric algebra $\mathbb{G}_{1,1}$ can be defined by

$$\mathbb{G}_{1,1} := gen\{\mathbf{c}_1, \mathbf{c}_2\}_{\mathbb{R}} = span\{1, \mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_1 \wedge \mathbf{c}_2\}, \tag{9}$$

  where $\mathbf{c}_1 := \frac{1}{2}(\mathbf{e}+\mathbf{f})$, $\mathbf{c}_2 := \frac{1}{2}(\mathbf{e}-\mathbf{f})$ and $\mathbf{c}_1 \wedge \mathbf{c}_2 := \frac{1}{2}(\mathbf{c}_1\mathbf{c}_2 - \mathbf{c}_2\mathbf{c}_1) = \mathbf{fe}$.

A Multiplication Table for the correlated null vector building blocks of $\mathbb{G}_{1,1}$ is given below.

Table 1: Multiplication table.

	$\mathbf{c}_1$	$\mathbf{c}_2$	$\mathbf{c}_1\mathbf{c}_2$	$\mathbf{c}_2\mathbf{c}_1$
$\mathbf{c}_1$	0	$\mathbf{c}_1\mathbf{c}_2$	0	$\mathbf{c}_1$
$\mathbf{c}_2$	$\mathbf{c}_2\mathbf{c}_1$	0	$\mathbf{c}_2$	0
$\mathbf{c}_1\mathbf{c}_2$	$\mathbf{c}_1$	0	$\mathbf{c}_1\mathbf{c}_2$	0
$\mathbf{c}_2\mathbf{c}_1$	0	$\mathbf{c}_2$	0	$\mathbf{c}_2\mathbf{c}_1$

A *degenerate* geometric subalgebras of $\mathbb{G}_{1,1}$ is

$$^{dgt}\mathbb{G}_{1,0,1} := span\{1, \mathbf{c}_1, \mathbf{c}_1 \wedge \mathbf{c}_2\}_{\mathbb{R}} \subset \mathbb{G}_{1,1} = gen\{\mathbf{c}_1, \mathbf{c}_2\}_{\mathbb{R}}.$$

In this degenerate algebra the hyperbolic bivector $\mathbf{c}_1 \wedge \mathbf{c}_2$, which anticommutes with $\mathbf{c}_1$, is interpreted as a vector. We now construct two degenerate subalgebras of $\mathbb{G}_{1,2}$,

$$^{dgt}\mathbb{G}_{0,1,1} := gen\{1, \mathbf{c}_1, \mathbf{c}_2 - \mathbf{c}_3, \mathbf{c}_1(\mathbf{c}_2 - \mathbf{c}_3)\} \subset \mathbb{G}_{1,2} = gen\{\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3\}_{\mathbb{R}}.$$

and

$$^{dgt}\mathbb{G}_{0,0,2} := span\{1, \mathbf{c}_1, \mathbf{c}_1(\mathbf{c}_2 - \mathbf{c}_3)\} \subset \mathbb{G}_{1,2}.$$

The geometric algebras $\mathbb{G}_{1,1}$ and $\mathbb{G}_{1,2}$ are further discussed in the next Section.

1.3 Coordinate matrices of $\mathbb{G}_{1,1}$, $\mathbb{G}_{2,0}$, $\mathbb{G}_{1,2}$ and $\mathbb{G}_2(\mathbb{C})$

Multiplication Table 2 is for a pair of correlated null vectors $\mathbf{a}, \mathbf{b}$ satisfying

$$\mathbf{a} \cdot \mathbf{b} = \frac{1}{2} \quad \iff \quad 2\mathbf{a} \cdot \mathbf{b} = (\mathbf{ab} + \mathbf{ba}) = 1. \tag{10}$$

It can be easily checked that the relations (7) and (10), and the Multiplication

Table 2: Multiplication table.

	a	**b**	**ab**	**ba**
a	0	**ab**	0	**a**
b	**ba**	0	**b**	0
ab	**a**	0	**ab**	0
ba	0	**b**	0	**ba**

Tables 1 & 2 are identical when $\mathbf{c}_1 \to \mathbf{a}$ and $\mathbf{c}_2 \to \mathbf{b}$.

The geometric algebra $\mathbb{G}_{1,1}$ is isomorphic to the familiar 2×2 real matrix algebra $\mathcal{M}_2(\mathbb{R})$, which define the *coordinate matrices* of $\mathbb{G}_{1,1}$ with respect to the the *spectral basis*,

$$\begin{pmatrix} 1 \\ \mathbf{a} \end{pmatrix} \mathbf{ba} \begin{pmatrix} 1 & \mathbf{b} \end{pmatrix} = \begin{pmatrix} \mathbf{ba} \\ \mathbf{a} \end{pmatrix} \begin{pmatrix} \mathbf{ba} & \mathbf{b} \end{pmatrix} = \begin{pmatrix} \mathbf{ba} & \mathbf{b} \\ \mathbf{a} & \mathbf{ab} \end{pmatrix}. \tag{11}$$

The matrix $[g] = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \in \mathcal{M}_2(\mathbb{R})$ is said to be the *coordinate matrix* of the geometric number $g \in \mathbb{G}_{1,1}$,

$$g = \begin{pmatrix} \mathbf{ba} & \mathbf{a} \end{pmatrix} [g] \begin{pmatrix} \mathbf{ba} \\ \mathbf{b} \end{pmatrix} = g_{11}\mathbf{ba} + g_{12}\mathbf{b} + g_{21}\mathbf{a} + g_{22}\mathbf{ab}, \tag{12}$$

with respect to the spectral basis (11), [5, Chp.3,pp.45-64]. Note that

$$[\mathbf{a}] = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \ [\mathbf{b}] = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \ [\mathbf{ba}] = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \ [\mathbf{ab}] = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{13}$$

The *standard basis* of $\mathbb{G}_{1,1}$ is generated by two vectors $\mathbf{e} := \mathbf{a}+\mathbf{b}$, $\mathbf{f} := \mathbf{a}-\mathbf{b}$, which satisfy the rules

$$\mathbf{e}^2 = 1 = -\mathbf{f}^2, \quad \mathbf{ef} = -\mathbf{fe} = 2\mathbf{b} \wedge \mathbf{a}, \quad (\mathbf{ef})^2 = 4(\mathbf{b} \wedge \mathbf{a})^2 = 1, \tag{14}$$

as already noted in (8) in the previous section. See Figure 2 on page 11. Thus,

$$\mathbb{G}_{1,1} = gen\{\mathbf{e}, \mathbf{f}\}_{\mathbb{R}} = span\{1, \mathbf{e}, \mathbf{f}, \mathbf{ef}\}_{\mathbb{R}}. \tag{15}$$

The elements $\{\mathbf{e}, \mathbf{f}, \mathbf{ef}\}$ are mutually anticommutive. By reordering and reinterpreting the standard basis elements in (15), we get the standard basis of geometric algebra

$$\mathbb{G}_2 := span\{1, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_{12}\} \widetilde{=} span\{1, \mathbf{e}, \mathbf{ef}, \mathbf{f}\}_{\mathbb{R}}, \tag{16}$$

where in this case we are identifying **ef** to be a vector, and **f** to be a bivector. Whereas the geometric algebras $\mathbb{G}_{1,1}\widetilde{=}\mathbb{G}_2$ are isomorphic, we have gained a *new interpretation* in terms of null vectors.

Taking the geometric algebra, $\mathbb{G}_{1,1}(\mathbb{C}_j)\widetilde{=}\mathbb{G}_2(\mathbb{C}_j)$ over the complex numbers with $j=\sqrt{-1}$, the calculation

$$(\mathbf{a}+j\mathbf{b})^2=\mathbf{a}^2+2j\mathbf{a}\cdot\mathbf{b}+j^2\mathbf{b}^2=j,$$

shows that the inner product of $\mathbf{a}$ with the *imaginary null vector* $j\mathbf{b}$ is j, since $\mathbf{a}^2=\mathbf{b}^2=0$. On the other hand,

$$(\mathbf{a}+j\mathbf{b})(\mathbf{a}-j\mathbf{b})=j\mathbf{ba}-j\mathbf{ab}=2j(\mathbf{b}\wedge\mathbf{a}), \tag{17}$$

produces the imaginary bivector $2j(\mathbf{b}\wedge\mathbf{a})$. Whereas the real bivector **ef** in (14) has square $+1$, the imaginary bivector in (17) has square -1.

Let $\mathbf{u}:=\mathbf{ef}$ be the unit hyperbolic bivector, and

$$\mathbf{u}_\pm:=\mathbf{ba}=\frac{1}{2}\big(1\pm(\mathbf{a}+\mathbf{b})(\mathbf{a}-\mathbf{b})\big)=\frac{1}{2}(1\pm\mathbf{ef})=\frac{1}{2}(1\pm\mathbf{u}),$$

respectively. The spectral basis (11) can be written in the form

$$\begin{pmatrix}1\\ \mathbf{a}\end{pmatrix}\mathbf{ba}\begin{pmatrix}1 & \mathbf{b}\end{pmatrix}=\begin{pmatrix}1\\ \mathbf{a}+\mathbf{b}\end{pmatrix}\mathbf{ba}\begin{pmatrix}1 & \mathbf{a}-\mathbf{b}\end{pmatrix}=\begin{pmatrix}\mathbf{u}_+ & \mathbf{eu}_-\\ \mathbf{eu}_+ & \mathbf{u}_-\end{pmatrix}.$$

Letting $\mathbf{e}_1=\mathbf{e}$ and $\mathbf{f}_1=\mathbf{f}$, the familiar Pauli matrices $\sigma_k:=[\mathbf{e}_k]$, for $k=1,2,3$, are

$$[\mathbf{e}_1]=\begin{pmatrix}0&1\\1&0\end{pmatrix},\quad [\mathbf{e}_2]:=[j\mathbf{f}_1]=\begin{pmatrix}0&-j\\j&0\end{pmatrix}\quad\text{and}\quad[\mathbf{e}_3]=\begin{pmatrix}1&0\\0&-1\end{pmatrix}, \tag{18}$$

where $\mathbf{f}_1=\mathbf{e}_1\mathbf{e}_3=-i\mathbf{e}_2$ and $i:=\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$ is the unit trivector of $\mathbb{G}_3$. The real geometric algebra $\mathbb{G}_{1,1}$ is a subalgebra of $\mathbb{G}_3$,

$$\mathbb{G}_{1,1}:=gen_{\mathbb{R}}\{\mathbf{e}_1,\mathbf{f}_1\}=gen_{\mathbb{R}}\{\mathbf{e}_1,\mathbf{e}_1\mathbf{e}_3\}\subset\mathbb{G}_3, \tag{19}$$

and

$$\mathbb{G}_{1,1}(\mathbb{C}_j)\equiv\mathbb{G}_2(\mathbb{C}_j)\widetilde{=}\mathbb{G}_3\widetilde{=}\mathbb{G}_{1,3}^+\widetilde{=}Mat_2(\mathbb{C}_j). \tag{20}$$

Whereas $\mathbf{f}_1=\mathbf{e}_1\mathbf{e}_3$ in (19) is interpreted as a vector in $\mathbb{G}_{1,1}$, it is interpreted as the bivector $\mathbf{e}_1\mathbf{e}_3$ in $\mathbb{G}_3$. On the other hand, $\mathbf{e}_3=-i\mathbf{e}_1\mathbf{e}_2$ is interpreted as an imaginary bivector in $\mathbb{G}_{1,1}(\mathbb{C})=\mathbb{G}_2(\mathbb{C})$ and a vector in $\mathbb{G}_3$. A complex 2×2 matrix

$$[g]=\begin{pmatrix}g_{11}&g_{12}\\g_{21}&g_{22}\end{pmatrix}\in\mathcal{M}_2(\mathbb{C}),$$

with $\mathbf{e}_{123}\equiv i=j$, is the *coordinate matrix* of the geometric number $g\in\mathbb{G}_3$. The convention $i=j$ follows from sign convention used in the definition of the Pauli matrices (18). Using (12),

$$g=\begin{pmatrix}\mathbf{ba} & \mathbf{a}\end{pmatrix}[g]\begin{pmatrix}\mathbf{ba}\\ \mathbf{b}\end{pmatrix}=g_{11}\mathbf{ba}+g_{12}\mathbf{b}+g_{21}\mathbf{a}+g_{22}\mathbf{ab} \tag{21}$$

Whereas the matrix $[g]$ in (21) is a matrix over the complex numbers $\mathbb{C}_j$, the equations (21) and (12) are otherwise identical.

Consider now the geometric algebra defined by $\mathbb{G}_{1,2} := gen\{\mathbf{e}_1, \mathbf{f}_1, \mathbf{f}_2\}$, where $\mathbf{e}_1, \mathbf{f}_1, \mathbf{f}_2$ are anticommutative and satisfy

$$\mathbf{e}_1^2 = 1 = -\mathbf{f}_1^2 = -\mathbf{f}_2^2.$$

The vectors $\{\mathbf{e}_1, \mathbf{f}_1, \mathbf{f}_2\}$ are called the *standard basis* of $\mathbb{G}_{1,2}$. Alternatively, $\mathbb{G}_{1,2}$ can be defined by

$$\mathbb{G}_{1,2} := span\{1, \mathbf{e}_1, \mathbf{f}_1, \mathbf{f}_2, \} = gen\{\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3\},$$

where

$$\mathbf{c}_1 := \frac{1}{2}(\mathbf{e}_1 + \mathbf{f}_1),\ \mathbf{c}_2 := \frac{1}{2}(\mathbf{e}_1 - \mathbf{f}_1),\ \mathbf{c}_3 := \mathbf{e}_1 + \mathbf{f}_2.$$

For an element $g \in \mathbb{G}_{1,2}$, and its complex coordinate matrix $[g]$,

$$g = \begin{pmatrix} \mathbf{c}_2\mathbf{c}_1 & \mathbf{c}_1 \end{pmatrix} [g] \begin{pmatrix} \mathbf{c}_2\mathbf{c}_1 \\ \mathbf{c}_2 \end{pmatrix} = \begin{pmatrix} 1 & \mathbf{e}_1 \end{pmatrix} \mathbf{u}_+ [g] \begin{pmatrix} 1 \\ \mathbf{f}_1 \end{pmatrix}$$

$$= g_{11}\mathbf{u}_+ + g_{12}\mathbf{e}_1 u_- + g_{21}\mathbf{e}_1\mathbf{u}_+ + g_{22}\mathbf{u}_-. \qquad (22)$$

From (13), the coordinate matrices $[\mathbf{c}_1] = [\mathbf{a}]$ and $[\mathbf{c}_2] = [\mathbf{b}]$ are known. The coordinate matrices of $[\mathbf{f}_2]$ and $[\mathbf{c}_3] = [\mathbf{e}_1] + [\mathbf{f}_2]$ are

$$[\mathbf{f}_2] = -[\mathbf{f}_1][\mathbf{e}_2][\mathbf{e}_3] = \begin{pmatrix} j & 0 \\ 0 & -j \end{pmatrix} = j[\mathbf{e}_3] \quad \text{and} \quad [\mathbf{c}_3] = \begin{pmatrix} j & 1 \\ 1 & -j \end{pmatrix}. \qquad (23)$$

It can be checked that $i = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 = \mathbf{e}_1\mathbf{f}_1\mathbf{f}_2 = j$.

2 The Geometric Algebras $\mathbb{G}_{p,q}$

The geometric algebras $\mathbb{G}_{n,n}$ and $\mathbb{G}_{1,n}$ are sub-algebras of $\mathcal{U}_{\mathcal{F}}$, defined in terms of the basic null vector building blocks in $\mathcal{U}_{\mathcal{F}}^1$. Here, only the scalar fields $\mathbb{R}$ and $\mathbb{C}$ of real and complex numbers are considered. After the leisurely introduction already given, formal definitions and terminology are introduced and developed. Citations to proofs and more advanced material are provided.

2.1 The geometric algebras $\mathbb{G}_{n,n}$ and $\mathbb{G}_{1,n}$

Definition 1 *Two null vectors $\mathbf{c}_i, \mathbf{c}_j \in \mathcal{U}_{\mathcal{F}}^1$ are said to be correlated if*

$$\mathbf{c}_i \cdot \mathbf{c}_j := \frac{1}{2}(\mathbf{c}_i\mathbf{c}_j + \mathbf{c}_j\mathbf{c}_i) \neq 0.$$

Definition 2 *Two sets $\mathcal{A}_n, \mathcal{B}_n \subset \mathcal{U}_{\mathcal{F}}^1$ of anticommuting null-vectors $\mathcal{A}_n := \{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$, and $\mathcal{B}_n := \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$, generating the Grassmann algebras $\mathcal{G}_{2^n}(\mathcal{A}_n)$ and $\mathcal{G}_{2^n}(\mathcal{B}_n)$, are said to be globally dual if*

$$\mathbf{a}_i \cdot \mathbf{b}_j = \frac{1}{2}(\mathbf{a}_i\mathbf{b}_j + \mathbf{b}_j\mathbf{a}_i) = \frac{1}{2}\delta_{ij},$$

for each $1 \le i \le j \le n$.

The concept of global duality in $\mathbb{G}_{n,n}$ encompasses the concept of duality between a vector n-space and its dual n-space of covectors. It the has the significant advantage of being embedded in a comprehensive algebraic structure [2, p.3].

Definition 3 *A set $\mathcal{C}_{n+1} := \{\mathbf{c}_1, \dots, \mathbf{c}_{n+1}\} \subset \mathcal{U}^1_{\mathcal{F}}$ of $(n+1)$-correlated null-vectors, with $\mathbf{c}_i \cdot \mathbf{c}_j := \frac{1}{2}(1-\delta_{ij})$, is said to be locally dual.*

The concept of local duality was first defined and studied in [6, 7].

For globally dual Grassmann algebras $\mathcal{G}_{2^n}(\mathcal{A}_n)$, $\mathcal{G}_{2^n}(\mathcal{B}_n)$, we have

Theorem 1 *The 2^{2n}-dimensional geometric algebra*

$$\mathbb{G}_{n,n} := gen\{\mathcal{A}_n \cup \mathcal{B}_n\}_{\mathcal{F}} = gen\{\mathbf{e}_1, \dots, \mathbf{e}_n, \mathbf{f}_1, \dots, \mathbf{f}_n\} \subset \mathcal{U}_{\mathcal{F}},$$

where $\mathbf{e}_i := \mathbf{a}_i + \mathbf{b}_i$, $\mathbf{f}_i := \mathbf{a}_i - \mathbf{b}_i$, for $i = 1, \dots, n$.

The geometric algebras $\mathbb{G}_{n,n} \widetilde{=} Mat_{2^n}(\mathbb{R})$, [5, 6, pp.81-88].

The construction of a Grassmann algebra $\mathcal{G}_{2^{n+1}}(\mathcal{C}_{n+1})$, under the antisymmetric wedge product, together with the extra structure of *local duality*,

$$\mathbf{c}_i \cdot \mathbf{c}_j := \frac{1}{2}(\mathbf{c}_i\mathbf{c}_j + \mathbf{c}_j\mathbf{c}_i) = \frac{1}{2}(1-\delta_{ij}),$$

defines the geometric algebra $\mathbb{G}_{1,n}$. Local duality makes possible a new representation theory of geometric algebras not based wholly on the concept of a quadratic form [6, 7]. For $k \geq 1$, let $\mathbf{C}_k := \mathbf{c}_1 + \cdots + \mathbf{c}_k$ and $\wedge\mathcal{C}_k := \mathbf{c}_1 \wedge \cdots \wedge \mathbf{c}_k$.

Theorem 2 *The geometric algebra*

$$\mathbb{G}_{1,n} = gen\{\mathcal{C}_{n+1}\}_{\mathbb{R}} = gen\{\mathbf{e}_1, \mathbf{f}_1, \dots, \mathbf{f}_n\} \subset \mathcal{U}_{\mathbb{R}},$$

where $\mathbf{e}_1 := \mathbf{c}_1 + \mathbf{c}_2$ and $\mathbf{f}_1 := \mathbf{c}_1 - \mathbf{c}_2$. For $2 \leq k \leq n$, and $\alpha_k := \frac{\sqrt{2}}{\sqrt{k(k-1)}}$, $\mathbf{f}_k := \alpha_k\bigg((k-1)\mathbf{c}_{k+1} - \mathbf{C}_k\bigg)$. For $n \geq 1$, $\mathbf{e}_1\mathbf{f}_{1\cdots n} = -\frac{(\sqrt{2})^{n+1}}{\sqrt{n}} \wedge \mathcal{C}_{n+1}$.

The following **Theorem 3** provides a link between $\mathbb{G}_{p+1,q+1}$ and its matrix representation $Mat_2(\mathbb{G}_{p,q})$ in terms of a 2×2 matrix algebra over $\mathbb{G}_{p,q}$, [5, p.71]. Iterating this Theorem in the cases $p-q \geq 0$, $p-q=0$, or $p-q<0$, provides a new approach to the classification of all real and complex geometric algebras in terms of their metric free null vectors and isomorphic coordinate matrix algebras.

Theorem 3 $\mathbb{G}_{p+1,q+1} \widetilde{=} Mat_2(\mathbb{G}_{p,q})$.

Proof:

Recall that

$$\mathbb{G}_{p,q} = \mathbb{R}(\mathbf{e}_1, \dots, \mathbf{e}_p, \mathbf{f}_1, \dots, \mathbf{f}_q) \quad \text{and} \quad \mathbb{G}_{p+1,q+1} = \mathbb{R}(\mathbf{e}_1, \dots, \mathbf{e}_p, \mathbf{e}, \mathbf{f}_1, \dots, \mathbf{f}_q, \mathbf{f}).$$

Any element $G \in \mathbb{G}_{p+1,q+1}$ can be expressed in the form

$$G = g_0 + g_1\mathbf{e} + g_2\mathbf{f} + g_3\mathbf{ef}, \tag{24}$$

where $g_\mu \in \mathbb{G}_{p,q}$ for $0 \le \mu \le 3$. Applying the spectral basis (15) to $\mathbb{G}_{p+1,q+1}$, and again noting that

$$\begin{pmatrix} 1 & \mathbf{e} \end{pmatrix} u_+ \begin{pmatrix} 1 \\ \mathbf{e} \end{pmatrix} = u_+ + \mathbf{e}u_+\mathbf{e} = u_+ + u_- = 1,$$

we calculate

$$G = \begin{pmatrix} 1 & \mathbf{e} \end{pmatrix} u_+ \begin{pmatrix} 1 \\ \mathbf{e} \end{pmatrix} G \begin{pmatrix} 1 & \mathbf{e} \end{pmatrix} u_+ \begin{pmatrix} 1 \\ \mathbf{e} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & \mathbf{e} \end{pmatrix} u_+ \begin{pmatrix} G & G\mathbf{e} \\ \mathbf{e}G & \mathbf{e}G\mathbf{e} \end{pmatrix} u_+ \begin{pmatrix} 1 \\ \mathbf{e} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & \mathbf{e} \end{pmatrix} u_+ \begin{pmatrix} g_0 + g_3 & g_1 - g_2 \\ g_1^- + g_2^- & g_0^- - g_3^- \end{pmatrix} \begin{pmatrix} 1 \\ \mathbf{e} \end{pmatrix} = \begin{pmatrix} 1 & \mathbf{e} \end{pmatrix} u_+ [G] \begin{pmatrix} 1 \\ \mathbf{e} \end{pmatrix},$$

where

$$[G] := \begin{pmatrix} g_0 + g_3 & g_1 - g_2 \\ g_1^- + g_2^- & g_0^- - g_3^- \end{pmatrix} \in M_2(\mathbb{G}_{p,q}),$$

and $g^- := \mathbf{e}g\mathbf{e}$ is the operation of *geometric inversion* in $\mathbb{G}_{p,q}$ obtained by replacing all vectors in g by their negatives.

□

Corollary 1 *For $p = 0$ and $q = n$, $\mathbb{G}_{1,n+1} \widetilde{=} Mat_2(\mathbb{G}_{0,n})$. Similarly, for $p = n$ and $q = 0$, $\mathbb{G}_{n+1,1} \widetilde{=} Mat_2(\mathbb{G}_{n,0})$.*

Corollary 2 *For $p = n$ and $q = n$, $\mathbb{G}_{n,n} \widetilde{=} Mat_{2^n}(\mathbb{R})$.*

Corollary 3 *For $n = 1, 2, 3, 4, 5$,*

$$\mathbb{G}_{1,n} \widetilde{=} Mat_2(\mathbb{R}),\ Mat_2(\mathbb{C}),\ Mat_2(\mathbb{H}),\ Mat_2({}^2\mathbb{H}),$$

respectively. For $n = 6, 7$, $\mathbb{G}_{1,6} \widetilde{=} Mat_8(\mathbb{C})$, and $\mathbb{G}_{1,7} \widetilde{=} Mat_{16}(\mathbb{R})$.

Theorem 3, and its three **Corollaries**, offer an alternative approach to proving **Theorems 1** and **2**.

2.2 Double number fields

One of the main objective of this work is to demonstrate how the universal algebra $\mathcal{U}_\mathcal{F}$ of null vectors offers unexpected new insight into the structure of real and complex geometric algebras. In order to classify these geometric algebras in terms of more familiar isomorphic matrix algebras, *double number fields* are used. Let $\mathcal{F}$ be any number field. The double number field of $\mathcal{F}$ is

$$^2\mathcal{F} := \{(r,s)| \ \ r,s \in \mathcal{F}\}. \tag{25}$$

For $(r_1,s_1),(r_2,s_2) \in {}^2\mathcal{F}$, *double addition* and *double multiplication* are defined by

$$(r_1,s_1)+(r_2,s_2) = (r_1+r_2, s_1+s_2), \text{ and } (r_1,s_1)(r_2,s_2) = (r_1r_2, s_1s_2).$$

Double number fields for $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$ are ${}^2\mathbb{R} := (\mathbb{R},\mathbb{R})$, ${}^2\mathbb{C} := (\mathbb{C},\mathbb{C})$, ${}^2\mathbb{H} = (\mathbb{H},\mathbb{H})$ are used in the isomorphic coordinate matrix representation of the geometric algebras $\mathbb{G}_{p,q}$. Complex scalars $\mathbb{C}_j := \mathbb{R}(j)$ are defined by extending the real number system $\mathbb{R}$ to include an *imaginary unit* $j := \sqrt{-1}$. The hyperbolic numbers ${}^2\mathbb{R} := \mathbb{R}(I)$ is obtained by extending the real numbers $\mathbb{R}$ to include a *hyperbolic unit* I with the property that $I \notin \mathbb{R}$ and $I^2 = 1$, [5, 10].

Let us reexamine these ideas in the context of the null vector real number line $\mathbb{R}_\mathbf{c}$, pictured in Figure 1, which is the *oriented direction* of a ray of light on the 3-dimensional *lightcone* in 4-dimensional Minkowski spacetime $\mathbb{R}^{1,3}$. How each such ray is seen in the relative three dimensional *Euclidean rest frame* of each observer is explained in Section 4. Each null vector $\mathbf{c} \in \mathcal{U}_\mathcal{F}$ is an oriented direction with no assigned scale. The concept of a *scalar* is independent of the concept of *scale*, and the concept of *length* is independent of the concept of *direction*. Multiplication Table 1 implies that the concepts of global and local duality coincide in the geometric algebra $\mathbb{G}_{1,1} := gen\{\mathbf{c}_1, \mathbf{c}_2\}$ generated by two locally dual null vectors.

Historically, $j = \sqrt{-1}$ is identified with the *imaginary unit* on the vertical line in the *coordinate number plane* $\mathbb{R}^2$. The relation $\mathbf{e}^2 = 1$ suggests we identify the classical horizontal real number line $\mathbb{R}$ with vector number line $\mathbb{R}_\mathbf{e}$, constructed from $\mathbf{e}$. Similarly, the classical vertical imaginary number line is replaced by $\mathbb{R}_\mathbf{f}$, the *pseudo-vector* number line. Taken together, the correlated null vectors $\mathbf{c}_1, \mathbf{c}_2$, or alternatively, the horizontal and vertical vector lines $R_\mathbf{e}$ and $\mathbb{R}_\mathbf{f}$ define the geometric algebra

$$\mathbb{G}_{1,1} \widetilde{=} Mat_2(\mathbb{R}), \tag{26}$$

see Figure 2. We have the following relations,

$$\mathbb{G}_{1,0} \widetilde{=} {}^2\mathbb{R}, \quad \mathbb{G}_{1,0}(\mathbb{C}_j) \widetilde{=} {}^2\mathbb{C}, \quad \text{and} \quad \mathbb{G}_{0,2} \widetilde{=} \mathbb{H}, \tag{27}$$

[5, p. 75].

Analogous to the classical *complex number plane*,

$$\mathbb{C}_j := \{x + jy| \ \ x,y \in \mathbb{R}\},$$

we define the *complex null vector plane* of the correlated null vectors $\mathbf{c}_1$ and $\mathbf{c}_2$ to be

$$\mathbb{C}_j(\mathbf{c}_1,\mathbf{c}_2) := \{x\mathbf{c}_1 + jy\mathbf{c}_2| \ x,y \in \mathbb{R}\} \widetilde{=} \mathbb{G}_2(\mathbb{C}_j) \widetilde{=} \mathbb{G}_{1,2} \widetilde{=} \mathbb{G}_{3,0} \widetilde{=} Mat_2(\mathbb{C}_j). \tag{28}$$

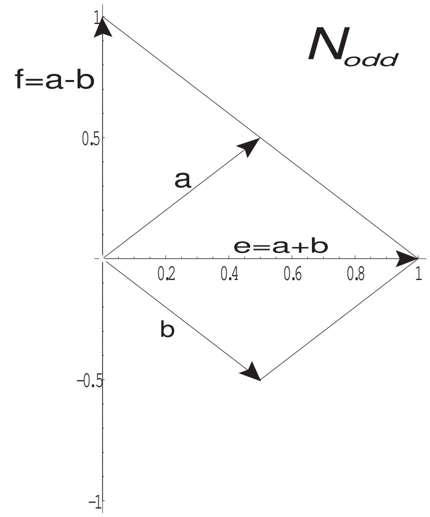

Figure 2: The null vector number plane $\mathbb{R}(\mathbf{a}, \mathbf{b})$ defined by the null vectors $\mathbf{a} = \mathbf{c}_1$ and $\mathbf{b} = \mathbf{c}_2$ in the geometric algebra $\mathbb{G}_{1,1}$.

For $Z = x\mathbf{c}_1 + jy\mathbf{c}_2$, $Z^2 = j = -\overline{Z}^2$, and

$$Z\overline{Z} = (x\mathbf{c}_1 + jy\mathbf{c}_2)(x\mathbf{c}_1 - yj\mathbf{c}_2) = 2xyj(\mathbf{c}_2 \wedge \mathbf{c}_1) = -\overline{Z}Z.$$

Contrast (28) with

$$\mathbb{G}_{1,2}(\mathbb{C}_j) \widetilde{=} \mathbb{G}_3(\mathbb{C}_j) \widetilde{=} Mat_2({}^2\mathbb{C}_j), \tag{29}$$

[5, p. 75]. The double double quaternions $\mathbb{G}_{0,3} \equiv {}^2\mathbb{H}$ are also necessary in the classification scheme of geometric algebras as isomorphic *coordinate matrix algebras.*

3 Classification of Geometric Algebras

With the **Three Theorems** in hand, the matrix representation of any geometric algebra $\mathbb{G}_{p,q}$ is reduced to the study of the representations of lower dimensional geometric algebras satisfying $p + q < 8$, known as the Cartan-Bott 8-periodicity relation. It follows that any real or complex geometric algebra can be represented by coordinate matrices over $\mathbb{R}, \mathbb{C}, \mathbb{H}$, ${}^2\mathbb{R}$, ${}^2\mathbb{C}$, ${}^2\mathbb{H}$. From **Theorem 1** and **Theorem 2**, in the cases of $\mathbb{R}, \mathbb{C}, \mathbb{H}$ the geometric algebras are expressed uniquely in terms of their globally or locally defined dual null vectors. **Theorem 3** is used to show the same result is true for geometric algebras over the complex numbers.

3.1 Geometric algebras $\mathbb{G}_{2,1}, \mathbb{G}_3(\mathbb{C}), \mathbb{G}_{1,3}, \mathbb{G}_{3,1}$ and $\mathbb{G}_4(\mathbb{C})$

When $\mathbb{G}_{1,2}$ is complexified with $j := \sqrt{-1}$, the result is

$$\mathbb{G}_{1,2}(\mathbb{C}) \equiv \mathbb{G}_3(\mathbb{C}) \widetilde{=} Mat_2({}^2\mathbb{C}).$$

Note that the locally dual null basis $\{\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3\}$ can still be used, and the matrix representation employs a pair of two complex 2×2 matrices. In this case the imaginary unit cannot be identified with a real pseudoscalar element, as in (20).

The standard basis of the geometric algebra $\mathbb{G}_{1,3}$ is

$$\mathbb{G}_{1,3} := gen\{\mathbf{e}_1, \mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\}_{\mathbb{R}} \widetilde{=} \mathbb{G}_{0,4} \widetilde{=} \mathbb{G}_{4,0} \widetilde{=} Mat_2(\mathbb{H}) \widetilde{=}$$

$$span\{1, \mathbf{f}_1, \mathbf{e}_1\mathbf{f}_2, \mathbf{e}_1\mathbf{f}_{12}\}_{\mathbb{R}} + J\ span\{1, \mathbf{f}_1, \mathbf{e}_1\mathbf{f}_2, \mathbf{e}_1\mathbf{f}_{12}\}_{\mathbb{R}} \tag{30}$$

for $J = \mathbf{f}_{123}, J_{\pm} = (1 \pm J)$. To see that $\mathbb{G}_{0,2} \widetilde{=} \mathbb{H}$, note that $J^2 = 1$ and

$$\mathbb{H} = \mathbb{G}_{0,2} \widetilde{=} gen\{f_1, f_2\}_R = span\{1, \mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_{12}\} \widetilde{=} span\{1, \mathbf{f}_1, \mathbf{e}_1\mathbf{f}_2, \mathbf{e}_1\mathbf{f}_{12}\}_{\mathbb{R}}.$$

For the complex geometric algebra $\mathbb{G}_{1,3}(\mathbb{C}) \equiv \mathbb{G}_4(\mathbb{C})$, we have

$$\mathbb{G}_4(\mathbb{C}) \widetilde{=} \mathbb{G}_{4,1}(\mathbb{R}) \widetilde{=} \mathbb{G}_{2,3}(\mathbb{R}) \widetilde{=} \mathbb{G}_5(\mathbb{R}) \widetilde{=} Mat_4(\mathbb{C}). \tag{31}$$

Note that each of these real geometric algebras can be obtained by a reinterpretation of the elements in $\mathbb{G}_5(\mathbb{R})$, or any of the other isomorphic real algebras.

3.2 Geometric algebras $\mathbb{G}_5(\mathbb{C}), \mathbb{G}_6(\mathbb{C})$ and $\mathbb{G}_7(\mathbb{C})$

For $\mathbb{G}_5(\mathbb{C})$, we have for $p + q = 5$,

$$\mathbb{G}_5(\mathbb{C}) \equiv \mathbb{G}_{p,q}(\mathbb{C}) \widetilde{=} Mat_4({}^2\mathbb{C}). \tag{32}$$

For these cases $\mathbb{C}$ are the complex scalars with $j = \sqrt{-1}$.

For $p + q = 6$, $\mathbb{G}_6(\mathbb{C}) \equiv \mathbb{G}_{p+q}(\mathbb{C})$, and

$$\mathbb{G}_6(\mathbb{C}) \widetilde{=} \mathbb{G}_7(\mathbb{R}) \widetilde{=} \mathbb{G}_{5,2}(\mathbb{R}) \widetilde{=} \mathbb{G}_{3,4}(\mathbb{R}) \widetilde{=} \mathbb{G}_{1,6}(\mathbb{R}) \widetilde{=} Mat_8(\mathbb{C}). \tag{33}$$

For $\mathbb{G}_7(\mathbb{C})$, we have for $p + q = 7$

$$\mathbb{G}_7(\mathbb{C}) \equiv \mathbb{G}_{p,q}(\mathbb{C}) \widetilde{=} Mat_8({}^2\mathbb{C}). \tag{34}$$

For these cases $\mathbb{C}$ are the complex scalars with $j = \sqrt{-1}$.

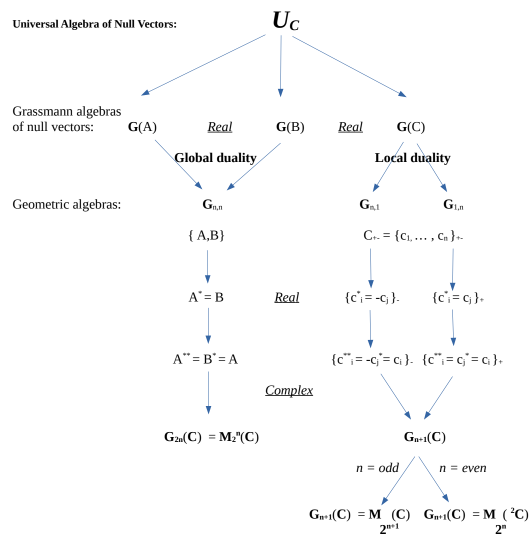

Figure 3: Global duality is equivalent to the classical concept of duality between a vector space and its dual covector space. The concept of local duality is constructed in Minkowski spacetime in the Lorentz geometric algebras $\mathbb{G}_{n,1}$ and $\mathbb{G}_{1,n}$. The double complex geometric algebras $^2\mathbb{C}$ only arise in the case of local duality.

4 Light Rays in Special Relativity

Minkowski light rays on the null cone in $\mathbb{G}_{1,3}$ are represented by oriented rays. But how do oriented rays appear to an observer in a given rest-frame? Every observer is defined by a positively oriented timelike Minkowski vector γ_0. WLOG, we now calculate the Hestenes *splitting* of a Minkowski null vector ray in the rest frame of $\gamma_0 = \mathbf{c}_1 + \mathbf{c}_2$, [1, 3, 8, 11, 12, 13].

Suppose that a null ray is given by

$$\mathbf{r} = \sum_{i=1}^{4} x_i \mathbf{c}_i \quad \text{where} \quad \mathbf{r}^2 = \sum_{1 \le i < j \le 4} x_i x_j$$

$$= x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4 = 0. \tag{35}$$

Using **Theorem 2**, in terms of the standard basis $\{e_1, f_1, f_2, f_3\}$ of $\mathbb{G}_{1,3}$,

$$\mathbf{r} = \frac{1}{2}(x_1 + x_2 + 2x_3 + 2x_4)\mathbf{e}_1 + \frac{1}{2}(x_1 - x_2)\mathbf{f}_1 + \frac{1}{2}(2x_3 + x_4)\mathbf{f}_2 + \frac{\sqrt{3}}{2}x_4\mathbf{f}_3.$$

The Hestenes splitting, with respect to a unit timelike Minkowski vector $\mathbf{e}_1 = \mathbf{C}_2$, is given by $\mathbf{r}\,\mathbf{C}_2 = \mathbf{r} \cdot \mathbf{C}_2 + \mathbf{r} \wedge \mathbf{C}_2$, where

$$\mathbf{r} \cdot \mathbf{C}_2 = \frac{1}{2}(x_1 + x_2) + (x_3 + x_4)$$

and

$$\mathbf{r} \wedge \mathbf{C}_2 = (x_1 - x_2)\mathbf{c}_1 \wedge \mathbf{c}_2 + x_3\mathbf{c}_3 \wedge (\mathbf{c}_1 + \mathbf{c}_2) + x_4\mathbf{c}_4 \wedge (\mathbf{c}_1 + \mathbf{c}_2)$$

$$= \frac{1}{2}(x_1 - x_2)\mathbf{E}_1 + \frac{1}{2}(2x_3 + x_4)\mathbf{E}_2 + \frac{\sqrt{3}}{2}x_4\mathbf{E}_3,$$

where $\mathbf{E}_1 := \mathbf{f}_1\mathbf{e}_1$, $\mathbf{E}_2 = \mathbf{f}_2\mathbf{e}_1$, $\mathbf{E}_3 = \mathbf{f}_3\mathbf{e}_1$ are three *orthonormal* relative Euclidean vectors of the observer $\mathbf{e}_1$.

Calculating

$$(\mathbf{r} \wedge \mathbf{C}_2)^2 = \frac{1}{4}(x_1 - x_2)^2 + \frac{1}{4}(2x_3 + x_4)^2 + \frac{3}{2}x_4^2,$$

and

$$(\mathbf{r} \cdot \mathbf{e}_1)^2 = \frac{1}{4}(x_1 + x_2 + 2x_3 + 2x_4)^2.$$

In the relative reference frame of each observer, the Minkowski metric on the lightcone takes the relative expression

$$(\mathbf{r} \cdot \mathbf{C}_2 + \mathbf{r} \wedge \mathbf{C}_2)(\mathbf{r} \cdot \mathbf{C}_2 - \mathbf{r} \wedge \mathbf{C}_2) = \mathbf{r}\,\mathbf{C}_2\mathbf{C}_2\,\mathbf{r} = \mathbf{r}^2 = 0.$$

Checking, we find that

$$(\mathbf{r} \cdot \mathbf{C}_2)^2 - (\mathbf{r} \wedge \mathbf{C}_2)^2 = \mathbf{r}^2 = 0,$$

as required by (35) for the oriented Minkowski ray $\mathbf{r}$.

Points $x, y \in \mathbb{G}_{1,3}^1$ are called *events* in spacetime. Given the *timelike* unit vector $\gamma_0 = \mathbf{c}_1 + \mathbf{c}_2$ of a Euclidean rest-frame,

$$x\gamma_0 = x \cdot \gamma_0 + x \wedge \gamma_0 = ct + \mathbf{x},$$

where $ct := x \cdot \gamma_0$ is the *relative time* and $\mathbf{x} := x \wedge x$ is the *relative position* of the event x as measured in the rest-frame of the observer. The relative *Euclidean distance* between the events x and y at the same relative time ct is

$$|\mathbf{y} - \mathbf{x}| := \sqrt{(\mathbf{y} - \mathbf{x})^2} = \sqrt{|\mathbf{y}|^2 + 2\mathbf{y} \cdot \mathbf{x} + |\mathbf{x}|^2}.$$

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