Neural Networks of Null Vectors in Geometric Algebra

Preprint · June 2025

DOI: 10.13140/RG.2.2.11829.18402

Garret Eugene Sobczyk
Universidad de las Américas Puebla

Neural Networks of Null Vectors in Geometric Algebra

Garret Sobczyk
Seminar on Fundamental Problems in Physics
Universidad de Las Americas-Puebla

June 24, 2025

Abstract

In the author's *Null Vector Number Systems* a *ray* or *null vector* in Lorentz-Minkowski spacetime is an *oriented direction*, unencumbered by a metric defined length. Researchers are exploring neural networks utilizing Clifford's geometric algebras of multivectors in the hope that the black box nature of such networks can be better understood by capturing underlying geometric features in the data. Usually the weights of neurons in each layer of a linear network are determined by the inner product of that neuron with each of the other neurons in that layer. However, by employing geometric algebra, we also have the outer product at our disposal. In addition to further developing the algebraic structure of a locally dual set of null vectors, we explore how geometric networks of affine neurons might be constructed.

0 Introduction

A real *null vector* represents an *oriented direction* of a *ray* of light. The concept of a null vector $\mathbf{v}$ is independent from the concept of a definite metric since $\mathbf{v}^2 = 0$. The (Clifford) geometric algebras $\mathbb{G}_{1,n}$ and $\mathbb{G}_{n,1}$ can be defined by *locally dual* sets of $n+1$ null vectors [21]. In [1], the authors demonstrate how geometrical ideas must be deeply intertwined with neural networks, however alchemistically their workings appear to be, see Figure 4. They further remark that while deep neural networks are locally simple they are globally complicated. Geometric algebras are brought into the picture in Mert Pilancei's paper [9]. In his ground breaking Stanford University YouTube Seminar [11], he examines the details of how the inner and outer products of vectors in geometric algebras can be utilized. In 2023, another critical milestone was achieved with the development of the neuro-network transformer [22]. Use of the transformer makes possible the parallelization of calculations, which had presented a serious limitation of what could be accomplished.

Many other papers offer new insights and methods into further developing the surprising successes of neural networks in artificial intelligence, robotics, computer vision, automatic language translation, and other areas. The mathematics of *null vector number systems*, developed in the geometric algebras $\mathbb{G}_{1,n}$ and $\mathbb{G}_{n,1}$, can be found in [17, 19, 18, 20]. The concept of a *null vector neuron*, set down in this work, might further simplify calculations and give new geometric insight into neural networks. T. Havel's *Heron's Formula in Higher Dimensions* provides new insight into how early Greek discoveries of properties of a triangle in the plane reflect much deeper corresponding relationships of simplices in higher dimensional spaces [4, 5]. E. Hitzer's, *On Symmetries of Geometric Algebra* discusses basic symmetry relationships in the geometric algebras $\mathbb{G}_{1,n}$ and $\mathbb{G}_{n,1}$, the geometric algebras of interest in this work. The paper [3] by C. Eur and M. Larson, *Combinatorial Hodge Theory*, discusses K ahler Packages, Polytopes, and Matroids, offer new ways of thinking about the structure of a neural network.

Section 1, gives a review of the universal algebra of null vectors constructed from Grassmann algebras of null vectors. A null vector characterizes the concept of an oriented direction unencumbered by a metrically defined length. Various basic identities in geometric algebra are reviewed.

Section 2, sets down the notation used, and studies properties of a locally dual null vector basis of Lorentz-Minkowski geometric algebras $\mathbb{G}_{1,n}$. Graphs of some of the basic functions are provided. A subsection is devoted to a concise treatment of the symmetric group acting on $\mathcal{S}_{n+1}$ defined by a product of $n+1$ null vectors.

Section 3, defines *local duality* in terms of inner products $\mathbf{C}_r \cdot \mathbf{C}_s$ between r- and s- dimensional blades of null vectors, used to construct dual *reciprocal bases.* A subsection constructs an isomorphism between the geometric algebra $\mathbb{G}_{n+1}$ of the Euclidean vector space $\mathbb{R}^{n+1}$ and the Lorentz-Minkowski spacetime algebra $\mathbb{G}_{1,n}$ of the pseudo-Euclidean vector space $\mathbb{R}^{1,n}$. The orthogonalization process has tremendous computational advantages, reducing calculations involving high dimension determinants to calculations involving simple addition formulas in a locally dual basis and its reciprocal.

Sections 4 and 5, suggest how a *Lorentz-Minkowski Neural Network* (LMNN) might be constructed using the new hierarchy of Lorentz-Minkowski geometric algebras and their unique properties. The isomorphism between the geometric algebras $\mathbb{G}_{n+1,0}$ and $\mathbb{G}_{1,n}$ offer new possibilities and insight into the nature of this inscrutable, even mysterious, new technology.

1 Universal algebra of null vectors

The *universal algebra of null vectors* $\mathcal{U}_\mathcal{F}$ over a field $\mathcal{F}$ is generated by a maximal set of null vectors $\mathcal{U}^1_\mathcal{F}$,

$$\mathcal{U}_\mathcal{F} := gen(\mathcal{U}^1_\mathcal{F}) = gen\{\mathbf{a}, \mathbf{b}, \mathbf{c}, \dots \mid 0 = \mathbf{a}^2 = \mathbf{b}^2 = \mathbf{c}^2 = ...\}_\mathcal{F}. \qquad (1)$$

Null vectors are added and multiplied according to the familiar rules of matrix algebra over a field $\mathcal{F}$, here chosen to be the real or complex numbers $\mathbb{R}$ or $\mathbb{C}$. A passing familiarity with the notation and ideas set down in [21] is recommended to the reader.

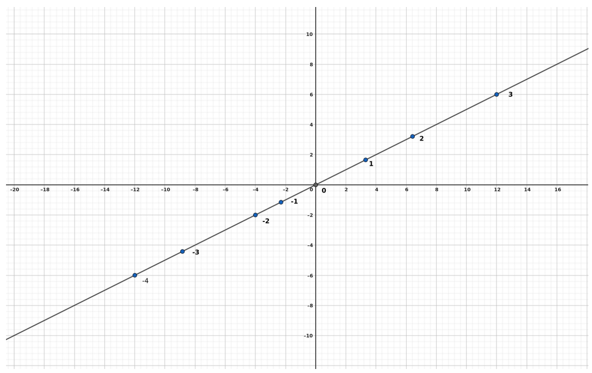

Figure 1: The concept of an oriented ray is independent of the concept of a definite metric. A *scale* which respects the natural ordering of the real numbers is shown. A scale defines the *orientation* and *origin* of the ray.

A null vector $\mathbf{a} \in \mathcal{U}_{\mathbb{R}}^1$ with $\mathbf{a}^2 = 0$ abstracts the direction of an *oriented ray* of light on the Einstein *light cone*, independent of the concept of its length. All rays on the light cone pass through a common point, the *origin* 0, and are seen by observers as straight lines in their rest frames, determined by their relative velocities. A *scaled* null vector $\mathbf{v}$ on the light cone is pictured in Figure 1. The vertex of the light cone is the origin of the coordinate system of spacetime. Rays on the light cone are said to be *regularly scaled* if all the rays are scaled as Euclidean real number lines passing through the origin, differing only in their directions.

Given two null vectors $\mathbf{a}, \mathbf{b} \in \mathcal{U}_1^1$, their *geometric product*

$$\mathbf{ab} := \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \wedge \mathbf{b}, \tag{2}$$

where $\mathbf{a} \cdot \mathbf{b} = \frac{1}{2}(\mathbf{ab} + \mathbf{ba})$ is the *scaler valued* inner product, and $\mathbf{a} \wedge \mathbf{b} := \frac{1}{2}(\mathbf{ab} - \mathbf{ba})$ is the *bivector valued* outer product of $\mathbf{a}$ and $\mathbf{b}$, respectively. For $s, t \in \mathbb{R}$,

$$(s\mathbf{a} + t\mathbf{b})^2 = s^2\mathbf{a}^2 + 2st\,\mathbf{a} \cdot \mathbf{b} + t^2\mathbf{b}^2 = 2st\,\mathbf{a} \cdot \mathbf{b}. \tag{3}$$

Since $\mathbf{a}$ and $\mathbf{b}$ are null vectors, only the symmetric inner product survives. If $\mathbf{a} \cdot \mathbf{b} = 0$, they are *orthogonal*. In this case, only the antisymmetric bivector valued outer product of $\mathbf{ab}$ survives,

$$\mathbf{ab} = \mathbf{a} \wedge \mathbf{b} = -\mathbf{b} \wedge \mathbf{a} = -\mathbf{ba}.$$

Suppose that $\mathbf{a}^2 = \mathbf{b}^2 = 0$, and $\mathbf{a} \cdot \mathbf{b} = \frac{1}{2}$. Defining $\mathbf{e} = \mathbf{a} + \mathbf{b}$ and $\mathbf{f} = \mathbf{a} - \mathbf{b}$, it is not hard to show that

$$\mathbf{e}^2 = 1, \mathbf{f}^2 = -1, \quad \text{and} \quad \mathbf{e} \cdot \mathbf{f} = 0.$$

See Figure 2, [15].

A set

$$\mathcal{C}_{n+1} := \{\mathbf{c}_1, \ldots, \mathbf{c}_{n+1}\} \subset \mathcal{U}_{\mathbb{R}}^1 \tag{4}$$

of $n+1$ null vectors $\mathbf{c}_i$ are said to be *locally dual* if they obey the rule

$$2\,\mathbf{c}_i \cdot \mathbf{c}_j := (\mathbf{c}_i\mathbf{c}_j + \mathbf{c}_j\mathbf{c}_i) = (1 - \delta_{ij}) \iff \mathbf{c}_i \cdot \mathbf{c}_j = \begin{pmatrix} 0, & i = j \\ \frac{1}{2}, & i \neq j) \end{pmatrix}, \tag{5}$$

for all $1 \leq i \leq j \leq n+1$. Such a set of null vectors generate the real geometric algebra

$$\mathbb{G}_{1,n} := gen\{\mathbf{c}_1, \ldots, \mathbf{c}_{n+1}\}_{\mathbb{R}}, \tag{6}$$

[21]. If $\mathbf{a} \wedge \mathbf{b} = 0$ then $\mathbf{b} = \lambda\mathbf{a}$ for some $\lambda \in \mathbb{R}$. If $\mathbf{c}_j$ is in the positive or future lightcone, and $\mathbf{a}^2 = 0$, $\mathbf{c}_j \cdot \mathbf{a} > 0$ then $\mathbf{a}$ is also on the *future* light cone. If $\mathbf{c}_j \cdot \mathbf{a} < 0$ then $\mathbf{a}$ is on *past* light cone.

The Multiplication Table 1, given below for the locally dual basis of $\mathbb{G}_{1,n}$, shows that the geometric product is completely determined by the multiplication of any pair of locally dual basis null vectors $\mathbf{c}_i, \mathbf{c}_j \in \mathbb{G}_{1,n}^1$. Whereas the inner product of a distinct pair, $\mathbf{c}_i \cdot \mathbf{c}_j = \frac{1}{2}$, the antisymmetric outer product $\mathbf{c}_i \wedge \mathbf{c}_j = -\mathbf{c}_j \wedge \mathbf{c}_i \in \mathbb{G}_{1,n}^2$ is bivector valued. Just as a null vector determines the oriented direction of a ray, the bivector $\mathbf{c}_i \wedge \mathbf{c}_j$ determines the direction of an oriented plane. The *Gramian* of $\mathbf{c}_i \wedge \mathbf{c}_j$ is defined by

$$(\mathbf{c}_i \wedge \mathbf{c}_j)^2 = -(\mathbf{c}_j \wedge \mathbf{c}_i) \cdot (\mathbf{c}_i \wedge \mathbf{c}_j) = -\begin{pmatrix} \mathbf{c}_i^2 & \mathbf{c}_i \cdot \mathbf{c}_j \\ \mathbf{c}_j \cdot \mathbf{c}_i & \mathbf{c}_j^2 \end{pmatrix} = -\det\begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} = \frac{1}{4}. \tag{7}$$

The *hyperbolic magnitude* of the bivector $\mathbf{c}_i \wedge \mathbf{c}_j$ is

$$|\mathbf{c}_i \wedge \mathbf{c}_j| := \sqrt{(\mathbf{c}_i \wedge \mathbf{c}_j)^2} = \frac{1}{2},$$

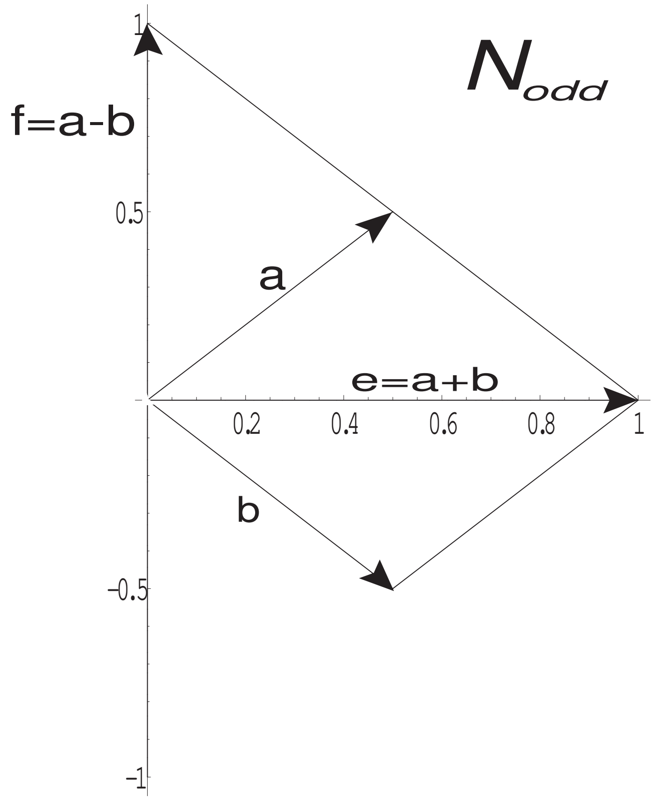

Figure 2: Given the null vectors $\mathbf{a}, \mathbf{b}$ with inner product $\frac{1}{2}$, the vectors $\mathbf{e}$ and $\mathbf{f}$ are orthogonal.

Table 1: Multiplication table.

	$\mathbf{c}_i$	$\mathbf{c}_j$	$\mathbf{c}_i\mathbf{c}_j$	$\mathbf{c}_j\mathbf{c}_i$
$\mathbf{c}_i$	0	$\mathbf{c}_i\mathbf{c}_j$	0	$\mathbf{c}_i$
$\mathbf{c}_j$	$\mathbf{c}_j\mathbf{c}_i$	0	$\mathbf{c}_j$	0
$\mathbf{c}_i\mathbf{c}_j$	$\mathbf{c}_i$	0	$\mathbf{c}_i\mathbf{c}_j$	0
$\mathbf{c}_j\mathbf{c}_i$	0	$\mathbf{c}_j$	0	$\mathbf{c}_j\mathbf{c}_i$

the same as the inner product $\mathbf{c}_i \cdot \mathbf{c}_j$.

The Gramian is a powerful tool giving new geometric insight into ancient Greek mathematics, and its generalization to higher dimensions [4]. In *hyperbolic geometry* the famous Euclidean trigonometric identity

$$r^2 = (x+iy)(x-iy) = r^2(\cos^2\theta + \sin^2\theta), \tag{8}$$

for $i^2 = -1, x = r\cos\theta, y = r\sin\theta$, becomes the corresponding hyperbolic trigonometric identity

$$\rho^2 = (x+uy)(x-uy) = \rho^2(\cosh^2\phi - \sinh^2\phi), \tag{9}$$

with $u := 2\mathbf{c}_i \wedge \mathbf{c}_j$ so that, $u^2 = +1$, [15].

For $k \geq 1$, the *k-blade* $\mathbf{C}_k := \mathbf{c}_1 \wedge \cdots \wedge \mathbf{c}_k$. Bivector and higher order blades, expressed in terms of the locally dual basis $\mathcal{C}_{k+1} = \{\mathbf{c}_1, \ldots, \mathbf{c}_{n+1}\} \in \mathbb{G}^1_{1,n}$, have many interesting properties. Given multivectors $\mathbf{M}, \mathbf{N} \in \mathbb{G}_{1,n}$, define $\mathbf{M} \otimes \mathbf{N} := \frac{1}{2}(\mathbf{MN} - \mathbf{NM})$. Some basic identities are given below:

1. $\mathbf{c}_1(\mathbf{c}_1 \wedge \mathbf{c}_2) = \mathbf{c}_1 \cdot (\mathbf{c}_1 \wedge \mathbf{c}_2) = -\frac{1}{2}\mathbf{c}_1$
2. $\mathbf{c}_1(\mathbf{c}_2 \wedge \mathbf{c}_3) = \mathbf{c}_1 \cdot (\mathbf{c}_2 \wedge \mathbf{c}_3) + \mathbf{c}_1 \wedge \mathbf{c}_2 \wedge \mathbf{c}_3 = \frac{1}{2}(\mathbf{c}_3 - \mathbf{c}_2) + \mathbf{c}_1 \wedge \mathbf{c}_2 \wedge \mathbf{c}_3$
3. $(\mathbf{c}_1 \wedge \mathbf{c}_2)(\mathbf{c}_1 \wedge \mathbf{c}_3) = (\mathbf{c}_{12} - \frac{1}{2})(\mathbf{c}_{13} - \frac{1}{2}) = \frac{1}{4} + \frac{1}{2}(\mathbf{c}_3 - \mathbf{c}_1)$.
4. $(\mathbf{c}_1 \wedge \mathbf{c}_2)(\mathbf{c}_3 \wedge \mathbf{c}_4) = (\mathbf{c}_1 \wedge \mathbf{c}_2) \otimes (\mathbf{c}_3 \wedge \mathbf{c}_4) + (\mathbf{c}_1 \wedge \mathbf{c}_2) \wedge (\mathbf{c}_3 \wedge \mathbf{c}_4)$
$$= \frac{1}{2}(\mathbf{c}_1 - \mathbf{c}_2)(\mathbf{c}_4 - \mathbf{c}_3) + \mathbf{C}_4$$
5. For $2 \leq j \leq k$, define $\mathbf{C}_{j,k} := (\mathbf{c}_{j+1} \wedge \cdots \wedge \mathbf{c}_{j+k})$. Then
$$\mathbf{C}_j\mathbf{C}_{j,k} := (\mathbf{c}_1 \wedge \cdots \wedge \mathbf{c}_j)(\mathbf{c}_{j+1} \wedge \cdots \wedge \mathbf{c}_{j+k})$$
$$= (\mathbf{c}_1 \wedge \cdots \wedge \mathbf{c}_j) \otimes (\mathbf{c}_{j+1} \wedge \cdots \mathbf{c}_{j+k}) + \mathbf{C}_{j+k}.$$

2 Locally Duality and the Symmetric Group

Let $\{\mathbf{c}_1, \ldots, \mathbf{c}_{n+1}\} \subset \mathcal{U}_{\mathbb{R}}$ be the locally dual basis (4) of the geometric algebra $\mathbb{G}_{1,n}$. Define $\mathbf{c}_{(0)} = \mathbf{c}_0 = 0$, $\mathbf{c}_{(k)} := \sum_{j=1}^k \mathbf{c}_j$, and $\mathbf{c}_{(j,k)} = \sum_{i=j+1}^{j+k} \mathbf{c}_i$. Then

$$\mathbf{c}_{(k)} = \sum_{j=1}^{k} \mathbf{c}_j \Longrightarrow \mathbf{c}^2_{(k)} = \frac{k(k-1)}{2}. \tag{10}$$

The following formulas are some of the many that can be derived.
Using (10), gives

$$\mathbf{c}_{(p,q)} = \mathbf{c}_{(p+q)} - \mathbf{c}_{(p)} = \sum_{j=p+1}^{p+q} \mathbf{c}_j \Longrightarrow \mathbf{c}^2_{(j,k)} = \mathbf{c}^2_{(q)}. \tag{11}$$

Using the right side of (11), gives

$$\mathbf{c}_{(p)} \cdot \mathbf{c}_{(p+q)} = \frac{1}{2}(\mathbf{c}^2_{(p+q)} + \mathbf{c}^2_{(p)} - \mathbf{c}^2_{(q)}) = \frac{p(p+q-1)}{2} \tag{12}$$

For $p \geq q$, let $p \to p-q$ in (12), gives

$$\mathbf{c}_{(p)} \cdot \mathbf{c}_{(p-q)} = \frac{(p-q)(p-1)}{2} = \mathbf{c}_{(p)}^2 - \frac{q(p-1)}{2} \tag{13}$$

$$\Big(\mathbf{c}_{(p)} \pm (\mathbf{c}_{(p+q)} - \mathbf{c}_{(p)})\Big)^2 = \frac{(p \pm q)^2 - (p+q)}{2}. \tag{14}$$

For $p \geq q$,

$$\mathbf{c}_{(p)} \cdot \mathbf{c}_{(q)} = \mathbf{c}_{(q)}^2 + \frac{(p-q)q}{2} = \frac{(p-1)q}{2}. \tag{15}$$

For $p \geq q$,

$$\begin{aligned}(\mathbf{c}_{(p)} \wedge \mathbf{c}_{(q)})^2 &= \frac{q(p-1)(p-q)}{4} \\ \implies \quad |\mathbf{c}_{(p)} \wedge \mathbf{c}_{(q)}| &= \frac{\sqrt{q(p-1)(p-q)}}{2}.\end{aligned} \tag{16}$$

See Figure 6, for a graph of the difference of equations (16) and (15). Using (15) and (16) gives the result

$$\frac{|\mathbf{c}_{(p)} \wedge \mathbf{c}_{(q)}|}{\mathbf{c}_{(p)} \cdot \mathbf{c}_{(q)}} = \sqrt{\frac{(p-q)}{q(p-1)}}. \tag{17}$$

Using the two parts of (14), gives

$$\frac{(\mathbf{c}_{(p)} - (\mathbf{c}_{(p+q)} - \mathbf{c}_{(p)}))^2}{\mathbf{c}_{(p+q)}^2} = \frac{(p-q)^2 - (p+q)}{(p+q)^2 - (p+q)}, \tag{18}$$

see Figure 5. Using (12) and (14), gives

$$\frac{\sqrt{\mathbf{c}_{(p)}^2 \mathbf{c}_{(q)}^2}}{\mathbf{c}_{(p+q)}^2} = \frac{\sqrt{p(p-1)q(q-1)}}{(p+q)(p+q-1)}, \tag{19}$$

see Figure 7.

For the vectors $\mathbf{c}_{(p)}, \mathbf{c}_{(q)} \in \mathbb{G}_{1,n}^1$, using (2) and (15),

$$\begin{aligned}\mathbf{c}_{(p)}^2 \mathbf{c}_{(q)}^2 = \mathbf{c}_{(p)} \mathbf{c}_{(q)} \mathbf{c}_{(q)} \mathbf{c}_{(p)} &= \Big(\mathbf{c}_{(p)} \cdot \mathbf{c}_{(q)} + \mathbf{c}_{(p)} \wedge \mathbf{c}_{(q)}\Big)\Big(\mathbf{c}_{(p)} \cdot \mathbf{c}_{(q)} - \mathbf{c}_{(p)} \wedge \mathbf{c}_{(q)}\Big) \\ &= (\mathbf{c}_{(p)} \cdot \mathbf{c}_{(q)})^2 - (\mathbf{c}_{(p)} \wedge \mathbf{c}_{(q)})^2,\end{aligned} \tag{20}$$

where in this case

$$\rho^2 = \mathbf{c}_{(p)}^2 \mathbf{c}_{(q)}^2 = \frac{1}{4} p(p-1)q(q-1), \ \cosh\phi = \frac{\mathbf{c}_{(p)} \cdot \mathbf{c}_{(q)}}{\sqrt{\mathbf{c}_{(p)}^2 \mathbf{c}_{(q)}^2}}, \sinh\phi = \frac{|\mathbf{c}_{(p)} \wedge \mathbf{c}_{(q)}|}{\sqrt{\mathbf{c}_{(p)}^2 \mathbf{c}_{(q)}^2}}.$$

2.1 The symmetric group $\mathcal{S}_{n+1}$

As before, let $\{\mathbf{c}_1, \cdots, \mathbf{c}_{n+1}\}$ be a locally dual null vector basis of $\mathbb{G}_{1,n} := gen\{\mathbf{c}_1, \ldots, \mathbf{c}_{n+1}\}_{\mathbb{R}}$. . In natural order let

$$\mathbf{S}_{n+1} = \mathbf{c}_1 \cdots \mathbf{c}_i \cdots \mathbf{c}_j \cdots \mathbf{c}_{n+1},$$

and let

$$(ij)\mathbf{S}_{n+1} := \mathbf{c}_1 \cdots \mathbf{c}_j \cdots \mathbf{c}_i \cdots \mathbf{c}_{n+1},$$

where only $\mathbf{c}_i$ and $\mathbf{c}_j$ are interchanged.

Theorem 1 *The 2-cycle (ij) acting on $\mathbf{S}_n$ gives*

$$(ij)\mathbf{S}_{n+1} = (-1)^{n+1}(\mathbf{c}_i - \mathbf{c}_j)\mathbf{S}_{n+1}(\mathbf{c}_i - \mathbf{c}_j)$$

Proof: Using the properties of local duality, we need only prove the theorem in locally in $\mathbf{S}_3 = \mathbf{c}_1\mathbf{c}_2\mathbf{c}_3$, and for $n > 2$ extend the argument to $\mathbf{S}_{n+1}$. In $\mathbf{S}_3$,

$$(\mathbf{c}_1 - \mathbf{c}_2)\mathbf{c}_1(\mathbf{c}_1 - \mathbf{c}_2) = \mathbf{c}_2, \quad (\mathbf{c}_1 - \mathbf{c}_2)\mathbf{c}_2(\mathbf{c}_1 - \mathbf{c}_2) = \mathbf{c}_1,$$

and $(\mathbf{c}_1 - \mathbf{c}_2)\mathbf{c}_3(\mathbf{c}_1 - \mathbf{c}_2) = -\mathbf{c}_3$. To avoid a notational nightmare, we complete the proof for $n = 3$ in $\mathbf{S}_{3+1} = \mathbf{c}_1\mathbf{c}_2\mathbf{c}_3\mathbf{c}_4$. For $\mathbf{c}_i, \mathbf{c}_j, \mathbf{c}_k \in \mathbf{S}_{n+1}$, with $\mathbf{c}_i \neq \mathbf{c}_j$,

$$(\mathbf{c}_i - \mathbf{c}_j)\mathbf{S}_4(\mathbf{c}_i - \mathbf{c}_j) = (\mathbf{c}_i - \mathbf{c}_j)\mathbf{c}_1\mathbf{c}_2\mathbf{c}_3\mathbf{c}_4(\mathbf{c}_i - \mathbf{c}_j) = (-1)^{3+1}$$

$$(\mathbf{c}_i - \mathbf{c}_j)\mathbf{c}_1(\mathbf{c}_i - \mathbf{c}_j)(\mathbf{c}_i - \mathbf{c}_j)\mathbf{c}_2(\mathbf{c}_i - \mathbf{c}_j)(\mathbf{c}_i - \mathbf{c}_j)\mathbf{c}_3(\mathbf{c}_i - \mathbf{c}_j)(\mathbf{c}_i - \mathbf{c}_j)\mathbf{c}_4(\mathbf{c}_i - \mathbf{c}_j).$$

If $k \neq \{i, j\}$, two signs are changed so the sign is positive, and the order of the null vectors in $\mathbf{S}_4$ is unchanged. If $k \in \{i, j\}$, two signs are again are changed so the sign is positive, and in this case $\mathbf{S}_4 \to (ij)\mathbf{S}_4$.

$\square$

The basic functions (10) - (20) can be extended to apply to the symmetric group. For example, since for $i > p$,

$$\mathbf{c}_{(p)}\mathbf{c}_i\mathbf{c}_{(p)} = (\mathbf{c}_1 + \cdots + \mathbf{c}_p)\mathbf{c}_i\mathbf{c}_{(p)} = (p - \mathbf{c}_i\mathbf{c}_{(p)})\mathbf{c}_{(p)} = p\mathbf{c}_{(p)} - \mathbf{c}_i\mathbf{c}_{(p)}^2, \tag{21}$$

which can be extended to give a formula for $\mathbf{c}_{(p)}\mathbf{S}_{n+1}\mathbf{c}_{(p)}$. It follows from (21) that

$$\mathbf{c}_{(p)}(\mathbf{c}_{(p+q)} - \mathbf{c}_{(q)})\mathbf{c}_{(p)} = pq\mathbf{c}_{(p)} - (\mathbf{c}_{(p+q)} - \mathbf{c}_{(q)})\mathbf{c}_{(p)}^2. \tag{22}$$

Another possibly useful formula is

$$\mathbf{c}_{(p+q)}\mathbf{c}_{(p)}\mathbf{c}_{(p+q)} = \mathbf{c}_{(p)}^3 + 2\mathbf{c}_{(p)}^2(\mathbf{c}_{(p+q)} - \mathbf{c}_{(p)}) + (\mathbf{c}_{(p+q)} - \mathbf{c}_{(p)})\mathbf{c}_{(p)}(\mathbf{c}_{(p+q)} - \mathbf{c}_{(p)}). \tag{23}$$

3 Local duality and Reciprocal Basis

Let $\{\mathbf{c}_1, \cdots, \mathbf{c}_{n+1}\}$ be a locally dual null vector basis of $\mathbb{G}_{1,n} := gen\{\mathbf{c}_1, \ldots, \mathbf{c}_{n+1}\}_{\mathbb{R}}$. Local duality has many surprising properties. Introducing the notation

$$\mathbf{C}_k := \mathbf{c}_1 \wedge \cdots \wedge \mathbf{c}_k, \tag{24}$$

the *pseudoscalar* of $\mathbb{G}_{1,n}$ is $\mathbf{C}_{n+1} = \mathbf{c}_1 \wedge \cdots \wedge \mathbf{c}_{n+1} \in \mathbb{G}_{1,n}^{n+1}$.

By *local duality* with respect to the locally dual basis $\{\mathbf{c}_1, \cdots, \mathbf{c}_{n+1}\}$, we mean any expression of the form $\mathbf{C}_r \cdot \mathbf{C}_s$. For example, recalling the definition of the Gramian determinant,

$$\mathbf{C}_r^2 = \mathbf{C}_r \cdot \mathbf{C}_r = \det \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & \cdots & 0 & \frac{1}{2} \\ & & \cdots & & \\ 0 & \frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2} \end{pmatrix}_r = \frac{r-1}{2^r}(-1)^{\frac{(r-1)(r-2)}{2}}. \tag{25}$$

For $r = 1, 2, 3, \ldots$,

$$\mathbf{C}_r^2 = \{0, \frac{1}{4}, -\frac{1}{4}, -\frac{3}{16}, \frac{1}{8}, \frac{5}{16}, -\frac{3}{16}, \frac{-7}{256}, \ldots\}$$

Even more interesting is the duality relations $\mathbf{C}_r \cdot \mathbf{C}_{r-1} \in \mathbb{G}_{1,n}^1$. For $r = 2, 3, 4, 5$, $\mathbf{C}_2 \cdot \mathbf{c}_1 = \frac{1}{2}\mathbf{c}_1$,

$$\mathbf{C}_3 \cdot \mathbf{C}_2 = \frac{1}{4}(\mathbf{c}_3 - \mathbf{c}_{(2)}), \mathbf{C}_4 \cdot \mathbf{C}_3 = \frac{1}{8}(2\mathbf{c}_4 - \mathbf{c}_{(3)}), \mathbf{C}_5 \cdot \mathbf{C}_4 = \frac{-1}{16}(2\mathbf{c}_5 - \mathbf{c}_{(4)}). \tag{26}$$

More generally for $r \geq 2$,

$$\mathbf{C}_r \cdot \mathbf{C}_{r-1} = (\mathbf{c}_1 \wedge \cdots \wedge \mathbf{c}_r) \cdot (\mathbf{c}_1 \wedge \cdots \wedge \mathbf{c}_{r-1}) = \frac{\mathbf{c}_{(r-1)} - (r-2)\mathbf{c}_r}{2^{r-1}}(-1)^{\frac{(r-1)(r-2)}{2}}. \tag{27}$$

The proof of (27) starts by showing

$$\mathbf{C}_n \mathbf{c}_{n+1} = \mathbf{C}_n \cdot \mathbf{c}_{n+1} + \mathbf{C}_{n+1},$$

and then expanding

$$\mathbf{C}_n \cdot \mathbf{c}_{n+1} = \frac{1}{2}(\mathbf{c}_1 - \mathbf{c}_2) \wedge (\mathbf{c}_2 - \mathbf{c}_3) \wedge \cdots \wedge (\mathbf{c}_{n-1} - \mathbf{c}_n)$$

$$= \frac{1}{2}(\mathbf{c}_1 - \mathbf{c}_2) \wedge (\mathbf{c}_2 - \mathbf{c}_3) \wedge \cdots \wedge (\mathbf{c}_{n-1} - \mathbf{c}_n) \in \mathbb{G}_{1,n}^{n-1}. \tag{28}$$

For $n = 2, 3$, $\mathbf{C}_2 \cdot \mathbf{c}_3 = \frac{1}{2}(\mathbf{c}_1 - \mathbf{c}_2)$ and $\mathbf{C}_3 \cdot \mathbf{c}_4 = \frac{1}{2}(\mathbf{c}_1 - \mathbf{c}_2) \wedge (\mathbf{c}_1 - \mathbf{c}_3)$, respectively.

A better way of approaching these identities is to start with the observation that for any $k \geq 1$,

$$\mathbf{C}_k \wedge \mathbf{c}_{(k)} = 0 \implies \mathbf{C}_k \mathbf{c}_{(k)} = \mathbf{C}_k \cdot \mathbf{c}_{(k)}. \tag{29}$$

Dotting $\mathbf{C}_k \wedge \mathbf{c}_{(k)} = 0$ on the right by $\mathbf{c}_{(k)}$ gives

$$\mathbf{C}_k = \frac{2}{k(k-1)}(\mathbf{C}_k \cdot \mathbf{c}_{(k)}) \wedge \mathbf{c}_{(k)}.$$

For $k \geq 2$,

$$(\mathbf{C}_k \cdot \mathbf{c}_1) \wedge \mathbf{c}_{(k)} = \frac{1}{k}\mathbf{C}_k,$$

and for $2 \leq r < k$,

$$(\mathbf{C}_k \cdot \mathbf{C}_r) \wedge \mathbf{c}_{(k)} = \frac{k-1}{2}\mathbf{C}_k \cdot [(\mathbf{c}_1 - \mathbf{c}_2) \wedge \cdots \wedge (\mathbf{c}_1 - \mathbf{c}_r)].$$

For $j, k > 1$, the vectors

$$\{\mathbf{c}_1 - \mathbf{c}_2, \mathbf{c}_1 - \mathbf{c}_3, \ldots, \mathbf{c}_1 - \mathbf{c}_n\} \in \mathbb{G}^1_{1,n}, \tag{30}$$

form a basis for the subalgebra $\mathbb{G}_{0,n} \subset \mathbb{G}_{1,n}$, and has the property

$$(\mathbf{c}_1 - \mathbf{c}_j) \cdot (\mathbf{c}_1 - \mathbf{c}_k) = -1 + \frac{1}{2}(1 - \delta_{jk}) = \begin{pmatrix} -1 & \text{for } j = k \\ \frac{1}{2} & \text{for } j \neq k \end{pmatrix}. \tag{31}$$

Together with the vector $\mathbf{c}_{(n+1)} = \mathbf{c}_1 + \cdots + \mathbf{c}_{n+1}$ they form a basis for $\mathbb{G}_{1,n}$. Note also that $\mathbf{c}_{(n+1)} \cdot (\mathbf{c}_1 - \mathbf{c}_k) = 0$, so it is orthogonal and anticommutes with $\mathbf{c}_1 - \mathbf{c}_k$ for each $1 < k \leq n+1$. For example,

$$\mathbf{C}_3 \cdot [(\mathbf{c}_1 - \mathbf{c}_2) \wedge (\mathbf{c}_1 - \mathbf{c}_3)] = -\frac{1}{2}\{[\mathbf{C}_2 \cdot (\mathbf{c}_1 - \mathbf{c}_2)] \wedge \mathbf{c}_3\} \cdot (\mathbf{c}_1 - \mathbf{c}_3) = -\frac{1}{4}\mathbf{c}_{(3)},$$

as expected. This identity is easily generalized to

$$\mathbf{C}_k \cdot \Big((\mathbf{c}_1 - \mathbf{c}_2) \wedge \cdots \wedge (\mathbf{c}_1 - \mathbf{c}_k)\Big) = (-1)^k \frac{\mathbf{c}_{(k)}}{2^{k-1}}. \tag{32}$$

Dotting this last identity on the right by $\mathbf{c}_{(k)}$ gives

$$\mathbf{C}_k\Big((\mathbf{c}_1 - \mathbf{c}_2) \wedge \cdots \wedge (\mathbf{c}_1 - \mathbf{c}_k) \wedge \mathbf{c}_{(k)}\Big) = (-1)^k \frac{\mathbf{c}^2_{(k)}}{2^{k-1}} = (-1)^k \frac{k(k-1)}{2^k},$$

which should be compared with (25) as a check for the value of $\mathbf{C}^2_k$. With (27) in hand, and letting $2 \leq r \leq n+1$, we get the basis

$$\{\mathbf{c}_1,\ \mathbf{c}_{(2)} - \mathbf{c}_3,\ \mathbf{c}_{(3)} - 2\mathbf{c}_4,\ ,\ldots,\ \mathbf{c}_{(n)} - (n-1)\mathbf{c}_{n+1}\} \subset \mathbb{G}_{1,n} \tag{33}$$

of an n dimensional subspace of $\mathbb{G}_{1,n}$. Except for the first null vector $\mathbf{c}_1$, the basis is orthogonal and can be normalized. It can be completed to a basis of $\mathbb{G}_{1,n}$ by adding the vector $\mathbf{c}_{(n+1)}$.

Given the locally dual basis $\{\mathbf{c}_1, \cdots, \mathbf{c}_{n+1}\} \subset \mathbb{G}^1_{1,n}$, using (25), (27) and introducing the normalizing factor $h = \frac{2^{n+1}}{n}(-1)^{\frac{n(n-1)}{2}}$, we can easily find its reciprocal basis $\{\mathbf{c}^1, \cdots, \mathbf{c}^{n+1}\}$ satisfying $\mathbf{c}^i \cdot \mathbf{c}_j = \delta_{ij}$. For $r = n+1$ and $1 \le k \le n$,

$$\mathbf{c}^k := -h(\mathbf{c}_{n+1} - \mathbf{c}_k)\mathbf{C}_{n+1}(\mathbf{c}_{n+1} - \mathbf{c}_k)(\mathbf{c}_{n+1} - \mathbf{c}_k)\mathbf{C}_n(\mathbf{c}_{n+1} - \mathbf{c}_k) = -h\widetilde{\mathbf{C}}_{n+1}\widetilde{\mathbf{C}}_n, \tag{34}$$

where

$$\widetilde{\mathbf{C}}_{n+1} := (\mathbf{c}_{n+1} - \mathbf{c}_k)\mathbf{C}_{n+1}(\mathbf{c}_{n+1} - \mathbf{c}_k), \ \widetilde{\mathbf{C}}_n := (\mathbf{c}_{n+1} - \mathbf{c}_k)\mathbf{C}_n(\mathbf{c}_{n+1} - \mathbf{c}_k).$$

It follows that

$$\mathbf{c}^k = -h\widetilde{\mathbf{C}}_{n+1}\widetilde{\mathbf{C}}_n = h(\mathbf{c}_{n+1} - \mathbf{c}_k)\mathbf{C}_{n+1}\mathbf{C}_n(\mathbf{c}_{n+1} - \mathbf{c}_k)$$

$$= h(\mathbf{c}_{n+1} - \mathbf{c}_k)\mathbf{C}_{n+1} \cdot \mathbf{C}_n(\mathbf{c}_{n+1} - \mathbf{c}_k) = \frac{2}{n}\Big(\mathbf{c}_{(n)}|_{k \to n+1)} - (n-1)\mathbf{c}_k\Big). \tag{35}$$

The orthogonalization process in Lorentz-Minkowsk spacetime $\mathbb{R}^{1,n}$ has tremendous computational advantages, reducing calculations involving high dimension determinants to calculations involving simple addition formulas in a locally dual basis and its reciprocal.

3.1 Euclidean and Lorentz-Minkowski spacetimes

The geometric algebras $\mathbb{G}_{n+1,0}$, and $\mathbb{G}_{1,n}$ of Euclidean and Lorentz-Minkowski spacetimes, respectively, are intimately related. Indeed, understanding the metric structure of one immediately translates to the metric structure of the other, and their respectively matrix representations. In order to easily translate from one to the other in their respective geometric algebras, we introduce the generalized symmetric and antisymmetric geometric products.

Given two multivectors $\mathbf{M}, \mathbf{M}$ in any geometric algebra $\mathbb{G}_{p,q}$, the symmetric $\mathbf{M} \odot \mathbf{N}$ and antisymmetric $\mathbf{M} \otimes \mathbf{N}$ geometric products are naturally defined by

$$\mathbf{MN} = \frac{1}{2}(\mathbf{MN} + \mathbf{NM}) + \frac{1}{2}(\mathbf{MN} - \mathbf{NM}) = \mathbf{M} \odot \mathbf{N} + \mathbf{M} \otimes \mathbf{N}, \tag{36}$$

where $\mathbf{M} \odot \mathbf{N} := \frac{1}{2}(\mathbf{MN} + \mathbf{NM})$ and $\mathbf{M} \otimes \mathbf{N} := \frac{1}{2}(\mathbf{MN} - \mathbf{NM})$. These generalized geometric products are useful in understanding the metric structure of geometric algebras and the many isomorphisms between them.

The geometric algebras $\mathbb{G}_{1,n}$ and $\mathbb{G}_{1+n,0}$ are algebraically isomorphic. In the *standard basis* of anticommuting gamma vectors γ_μ, and Euclidean vectors $\mathbf{e}_k$,

$$\mathbb{G}_{1,n} := gen\{\gamma_0, \gamma_1, \ldots, \gamma_n\}_{\mathbb{R}}, \ G_{n+1,0} := gen\{\mathbf{e}_1, \ldots, \mathbf{e}_{n+1}\}_{\mathbb{R}},$$

where $\gamma_0^2 = 1 = -\gamma_k^2$ for $1 \le k \le n$, and $\mathbf{e}_i^2 = 1$ for $1 \le k \le n+1$. Noting that for $\gamma_{k0} := \gamma_k \gamma_0$,

$$\{\gamma_0, \gamma_{10}, \ldots, \gamma_{n0}\},$$

is a set of a vector together with bivectors in $\mathbb{G}_{1,n}$, all anticommuting, the isomorphism is established by mapping $\gamma_0 \to \mathbf{e}_1$ and $\gamma_{k0} \to \mathbf{e}_{k+1}$ for $k = 1, \ldots, n$.

As an example, the spacetime algebra $\mathbb{G}_{1,3}$ of the *Dirac gamma matrices*, and the Euclidean geometric algebra $\mathbb{G}_{4,0}$ are isomorphic. The anticommuting bivectors $\sigma_k := \gamma_{k0}$ define the geometric algebra $\mathbb{G}_3$ of *Pauli matrices*. For $\mathbf{x} = x_1\sigma_1 + x_2\sigma_2 + x_3\sigma_3$, and $\mathbf{y} = y_1\sigma_1 + y_2\sigma_2 + y_3\sigma_3$, the basic identity (2) takes the form

$$\mathbf{xy} = \mathbf{x} \odot \mathbf{y} + \mathbf{x} \otimes \mathbf{y}$$

in the even subalgebra $\mathbb{G}_{1,3}^+$ of $\mathbb{G}_{1,3}$.

For the null vector $\mathbf{c}_1 \in \mathbb{G}_{1,n}^1$, define the set $\mathcal{E}_n$ of n bivectors

$$\mathcal{E}_n := \{\mathbf{E}_k = 2\mathbf{c}_1 \wedge \mathbf{c}_k | \ k = 2, \ldots, n+1\}, \tag{37}$$

with the same symbol used for the algebra $\mathcal{E}_n = gen\{\mathbf{E}_2, \ldots \mathbf{E}_{n+1}\}_{\mathbb{R}}$. It follows that

$$\mathcal{E}_n \subset \mathbb{G}_{1,n}^+ \subset \mathbb{G}_{1,n}. \tag{38}$$

Each such bivector $\mathbf{E}_k$ satisfies $\mathbf{E}_k = 2\mathbf{c}_1\mathbf{c}_k - 1$, and $\mathbf{E}_k^2 = \mathbf{E}_k \cdot \mathbf{E}_k = 1$. In addition, $\mathbf{E}_k \cdot \mathbf{E}_j = 1$ for $j \neq k$. Now successively calculate

$$\mathbf{E}_{23} := \mathbf{E}_2\mathbf{E}_3 = (2\mathbf{c}_{12} - 1)(2\mathbf{c}_{13} - 1) = 1 + 4\mathbf{c}_{13} - 2(\mathbf{c}_{12} + \mathbf{c}_{13})$$

$$= 1 + 2\mathbf{c}_1(-\mathbf{c}_2 + \mathbf{c}_3) = 1 + (-\mathbf{E}_2 + \mathbf{E}_3).$$

$$\mathbf{E}_{234} = \Big(1 + 2\mathbf{c}_1(-\mathbf{c}_2 + \mathbf{c}_3)\Big)(2\mathbf{c}_{14} - 1) = -1 + 2\mathbf{c}_1(\mathbf{c}_2 - \mathbf{c}_3 + \mathbf{c}_4) = (\mathbf{E}_2 - \mathbf{E}_3 + \mathbf{E}_4).$$

$$\mathbf{E}_{2345} = \Big(1 + 2\mathbf{c}_1(-\mathbf{c}_2 + \mathbf{c}_3 - \mathbf{c}_4 + \mathbf{c}_5)\Big) = 1 + (-\mathbf{E}_2 + \mathbf{E}_3 - \mathbf{E}_4 + \mathbf{E}_5).$$

More generally

Theorem 2 *1) For odd k,*

$$\mathbf{E}_{2\cdots k+1} = -1 + 2\mathbf{c}_1 \sum_{j=0}^{k-1} (-1)^j \mathbf{c}_{j+2} = \sum_{j=0}^{k-1} (-1)^j \mathbf{E}_{j+2}.$$

2) For even $k > 0$,

$$\mathbf{E}_{1\cdots k+1} = 1 + 2\mathbf{c}_1 \sum_{j=1}^{k} (-1)^j \mathbf{c}_{j+1} = 1 + \sum_{j=1}^{k} (-1)^j \mathbf{E}_{j+1}.$$

Since the dimension of $\mathcal{E}_n$, defined in (37), is $\begin{pmatrix} n+1 \\ 2 \end{pmatrix} = \frac{(n+1)n}{2}$, one might think that $\mathcal{E}_n$ is the Lie algebra $so(1,n)$ of bivectors of $\mathbb{G}_{1,n}$. However, this is not the case since $\mathbf{E}_2 \otimes \mathbf{E}_3 = \mathbf{E}_3 - \mathbf{E}_2$, and for $k > 2$

$$(\mathbf{E}_2 \otimes \mathbf{E}_3) \otimes \mathbf{E}_{k+1} = (\mathbf{E}_3 - \mathbf{E}_2) \otimes \mathbf{E}_{k+1} = (\mathbf{E}_{k+1} - \mathbf{E}_3) - (\mathbf{E}_{k+1} - \mathbf{E}_2) = \mathbf{E}_2 - \mathbf{E}_3.$$

This shows that each such $\mathbf{E}_k$ is an eigenbivector of $\mathbf{E}_2 \otimes \mathbf{E}_3$, so that the bivectors in $\mathcal{E}_n$ do not cover the Lie algebra $so_{1,n}$. However, by extending $\mathcal{E}_n(\otimes)$ to $\mathcal{E}_{n+1}(\otimes)$ by including the bivector $\mathbf{c}_2 \wedge \mathbf{c}_3$, we get the Lie algebra $so_{1,n}(\otimes) \subset \mathbb{G}_{1,n}^2$ of bivectors under the antisymmetric product $\otimes$.

Corollary 1 *The bivectors in $\mathcal{E}_{n+1}(\otimes)$ of the Lorentz group of orthogonal transformation in $\mathbb{G}_{1,n}$ is isomorphic to the Lie algebra $so_{1,n}(\otimes)$. We have*

$$\mathcal{E}_{n+1}(\otimes) \cong so_{1,n}(\otimes) \subset \mathbb{G}_{1,n}^2 \subset \mathbb{G}_{1,n}^+. \tag{39}$$

The above theorem tells us that the product $\mathbf{E}_{1\cdots k}$ of the bivectors $\mathbf{E}_j \in \mathbb{G}_{1,n}^2$ is a bivector blade when k is odd, and the sum of 1 plus a bivector blade when k is even. Importantly, in both of these cases the coordinate matrices of these quantities are all diagonal, and can be expressed in an alternating series of their sums.

4 Null vector neural networks

Null vector neural networks can be built from a set of locally dual null vectors $\{\mathbf{c}_1, \mathbf{c}_2, \ldots\}$ satisfying (5), and the Multiplication Table 1. A generic picture of a neural network is shown in Figure 3. Since geometric multiplication is defined locally, the size of the network can be expanded without modification of earlier versions. The total size of the network is determined by the total number of null vector neurons used in defining all of the layers of the network. At each stage of development, the number of null vector neurons used determines the size of the matrix representation of the network. If $n+1$ neurons are used, they are the locally dual null vector basis of the geometric algebra $\mathbb{G}_{1,n}$, which determines its matrix representation. Properties of a hypothetical Lorentz-Minkowski Neural Network (LMNN) of null vectors are summarized below.

1 Each neuron is represented by a null vector bias $\mathbf{c}_i \in \mathbb{G}_{1,n}^1$, $\mathbf{c}_i^2 = 0$. A second null vector $\mathbf{b}_i$ is introduced to capture non-linearity.

2 The untrained inner and outer products of all pairs $\mathbf{c}_i, \mathbf{c}_j$ of neurons satisfy

$$\mathbf{c}_i \cdot \mathbf{c}_j = |\mathbf{c}_i \wedge \mathbf{c}_j| = \frac{1}{2}(1 - \delta_{ij}),$$

where $\mathbf{c}_i \cdot \mathbf{c}_j := \frac{1}{2}(\mathbf{c}_i\mathbf{c}_j + \mathbf{c}_j\mathbf{c}_i)$, and $\mathbf{c}_i \wedge \mathbf{c}_j := \frac{1}{2}(\mathbf{c}_i\mathbf{c}_j - \mathbf{c}_j\mathbf{c}_i)$.

3 The geometric product of any two neurons is given in Multiplication Table 1, and $\mathbf{c}_i\mathbf{c}_j = \mathbf{c}_i \cdot \mathbf{c}_j + \mathbf{c}_i \wedge \mathbf{c}_j$.

4 In the training a LMNN, each layer of neurons updates inner and outer products from previous layers.

5 Inner and outer products of null vector neurons, and their biases when trained, capture geometric relationships hidden in the data.

6 A LMNN introduces a different hierarchy to the standard Euclidean based neural network, but fully incorporates the latter.

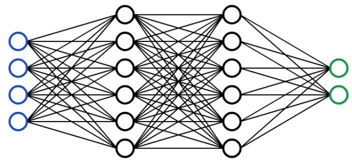

Figure 3: Generic neural network depicting input, layers, and output

.

7 LMNN, constructed from the geometric algebra $\mathbb{G}_{1,n}$, generalizes the standard Euclidean approach. The even subalgebra $\mathbb{G}_{1,n}^{+}$ of $G_{1,n}$ contains the Euclidean GA $\mathbb{G}_n$ built up from a set of n anti-commuting hyperbolic bivectors in $\mathbb{G}_{1,n}^{+}$.

8 The geometric product of neurons provides a new geometric measure of the content in data in forward passes or back propagation.

9 Traditionally, the inner product between neurons defines the activation weights between neurons. The inner and outer products together offer a more powerful geometric weight between neurons.

5 Future research on LMNN

- Explore how unique properties of $\mathbb{G}_{1,n}$ impact training and optimization of a LMNN.
- Explore the performance of LMNNs on standard data sets in comparison to the performance of other neural network models.
- Is the structure of a neuron in LMNN closer to the biological stucture of a neuron in the human or animal brains?

- Can a more direct comparison be made between the cognitive processes and neural function in a model LMNN with that in a biological neural network?
- Does local duality in hyperbolic Lorentz $\mathbb{G}_{1,n}$ make possible a more modular structure of a neural network than in Euclidean $\mathbb{G}_{n+1,0}$?
- Does local duality clarify how a LMNN model captures inherent geometrical features in the data?
- Explore how the geometric product between null vector neurons offers a new geometric measure of the geometric content in diverse data sets.
- Explore how the unique properties of $\mathbb{G}_{1,n}$ impacts the training and optimization of a LMNN.
- Explore how how the structure of a graph in [16] might be used to advantage in an LMNN.
- Explore the properties of geometric inner and outer product weights between neurons with the standard scalar weights.

References

[1] R. Balestriero, A.I. Humayun, R.G. Baraniuk, *On the Geometry of Deep Learning*, NOTICES OF THE AMERICAN MATHEMATICAL SOCIETY, Vol. 72, No. 4, (2025).

[2] E. van den Berg, *Some Insights into the Geometry and Training of Neural Networks*, arXiv:1605.00329v1 [cs.LG] 2 May 2016

[3] C. Eur, M. Larson, *Combinatorial Hodge Theory*, NOTICES OF THE AMERICAN MATHEMATICAL SOCIETY, Vol. 72, No. 3, (2025).

[4] T.F. Havel, *Heron's Formula in Higher Dimensions*, Advances in Applied Clifford Algebras, Vol. 34, No. 9, pp. 1-30, Springer Nature (2024).

[5] T.F. Havel, G. Sobczyk *The Zeros of Heron's Formula in Orthocentric Tetrahedra*, Conference: Applied Geometric Algebras in Computer Science and Engineering 2024At: Amsterdam, August 2024, DOI: 10.13140/RG.2.2.17531.94240/3

[6] E. Hitzer, *On Symmetries of Geometric Algebra $Cl(3,1)$ for Space-Time*, Adv. Appl. Clifford Algebras (2024) 34:30. https://diu,irg/10.10007/s00006-024-01331-0

[7] Qinyu Li, Yee Whye Teh, Razvan Pascanu, No Prop Training Neural Networks without Back-Propagation or Forward-Propagation, arXiv:2503.24322v1 [cs.LG] 31 Mar 2025

[8] D. Ruhe, J. K. Gupta, S. de Keninck, M. Welling, J. Brandstetter, *Geometric Clifford Algebra Networks*, Proceedings of the 40th International Conference on Machine Learning, Honolulu, Hawaii, USA. PMLR 202, 2023.

[9] M. Pilancei, *From Complexity to Clarity: Analytical Expressions of Deep Neural Network Weights via Clifford Algebra and Convexity*, Department of Electrical Engineering, Stanford University, pilanci@stanford.edu

[10] Shen Nie, Fengqi Zhu, Zebin, Xiaolu Zhang, Jingyang, Jun Hu, Jun Zhou, Yankai Lin, Ji-Rong Wen, Chongxuan Li, *Large Language Diffusion Models*, arXiv:2502.09992v1 [cs.CL] 14 Feb 2025

[11] M. Pilancei, *Hidden Convexity of Deep Neural Networks* https://www.youtube.com/watch?v=w-IotZ6RXK0 (2024).

[12] Y. Wang, S. Kim, P. Chu, I. Subramaniam, M. Pilanci, *Randomized Geometric Algebra Methods for Convex Neural Networks*, arXiv:2406.02806v2 [cs.LG] 8 Jun 2024

[13] J. Zhong, W. Cao, *Graph Geometric Algebra networks for graph representation learning*, Scientific Reports, (2025) 15:170 — https://doi.org/10.1038/s41598-024-84483-0

[14] V.G. Satorras, E. Hoogeboom, M. Welling, *E(n) Equivariant Graph Neural Networks*, arXiv:2102.09844v3 [cs.LG] 16 Feb 2022

[15] G. Sobczyk, *The Hyperbolic Number Plane*, American Mathematical Monthly, Vol. 26, pp. 268-280, 1995. https://doi.org/10.1080/07468342.1995.11973712

[16] G. Sobczyk, *Geometric Algebras of Light Cone Projective Graph Geometries*, Topical Collection for the First International Conference on Advanced Conputational Applications of Geometric Algebra (ICACGA) held in Denver, Colorado, USA, 2-5 October 2022.

[17] G. Sobczyk, *Matrix Gateway to Geometric Algebra, Spacetime and Spinors,* Independent Publisher November 2019. https://www.garretstar.com

[18] G. Sobczyk, *Global and Local Nilpotent Bases of Matrices*, Arxiv, September 2023. https://arxiv.org/pdf/2309.13084

[19] G. Sobczyk, *What's in a Pauli Matrix?* https://arxiv.org/abs/1905.08102v1

[20] G. Sobczyk, *Conformal Mappings in Geometric Algebra*, Notices of the AMS, Volume 59, Number 2, p.264-273, 2012.

[21] G. Sobczyk, *Null Vector Number Systems*, ResearchGate, DOI: 10.13140/RG.2.2.21400.92161, (2025).

[22] A. Vaswani, N. Shazeer, N. Parmanr, J. Uszkoreit, L. Jones, A. Gomez, L Kaiser, *Attention Is All You Need*, arXiv:1706.03762v7 [cs.CL] 2 Aug 2023

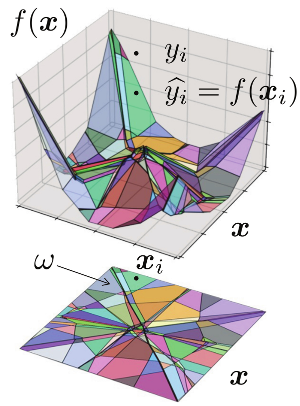

Figure 4: From Notices paper *On the Geometry of Deep Learning*, [1].

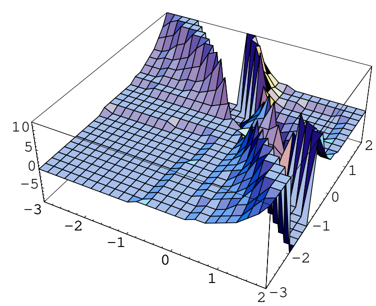

Figure 5: Graph of function defined in equation (18).

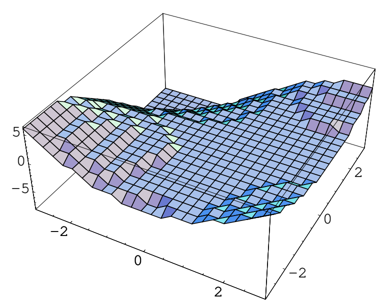

Figure 6: Graph of function defined by the difference $(\mathbf{c}_{(p)} \wedge \mathbf{c}_{(q)})^2 - \mathbf{c}_{(p)} \cdot \mathbf{c}_{(q)}$.

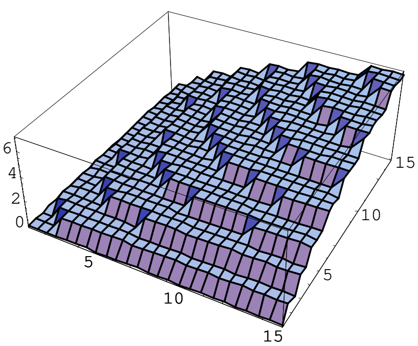

Figure 7: Graph of function defined in equation (19).