Laplace Equation on the Nullcone and Finsler Geometry

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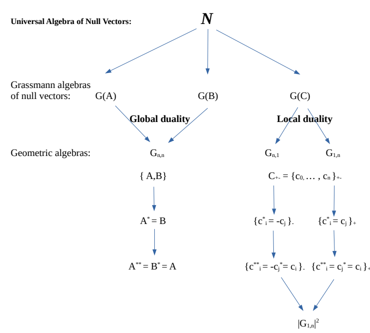
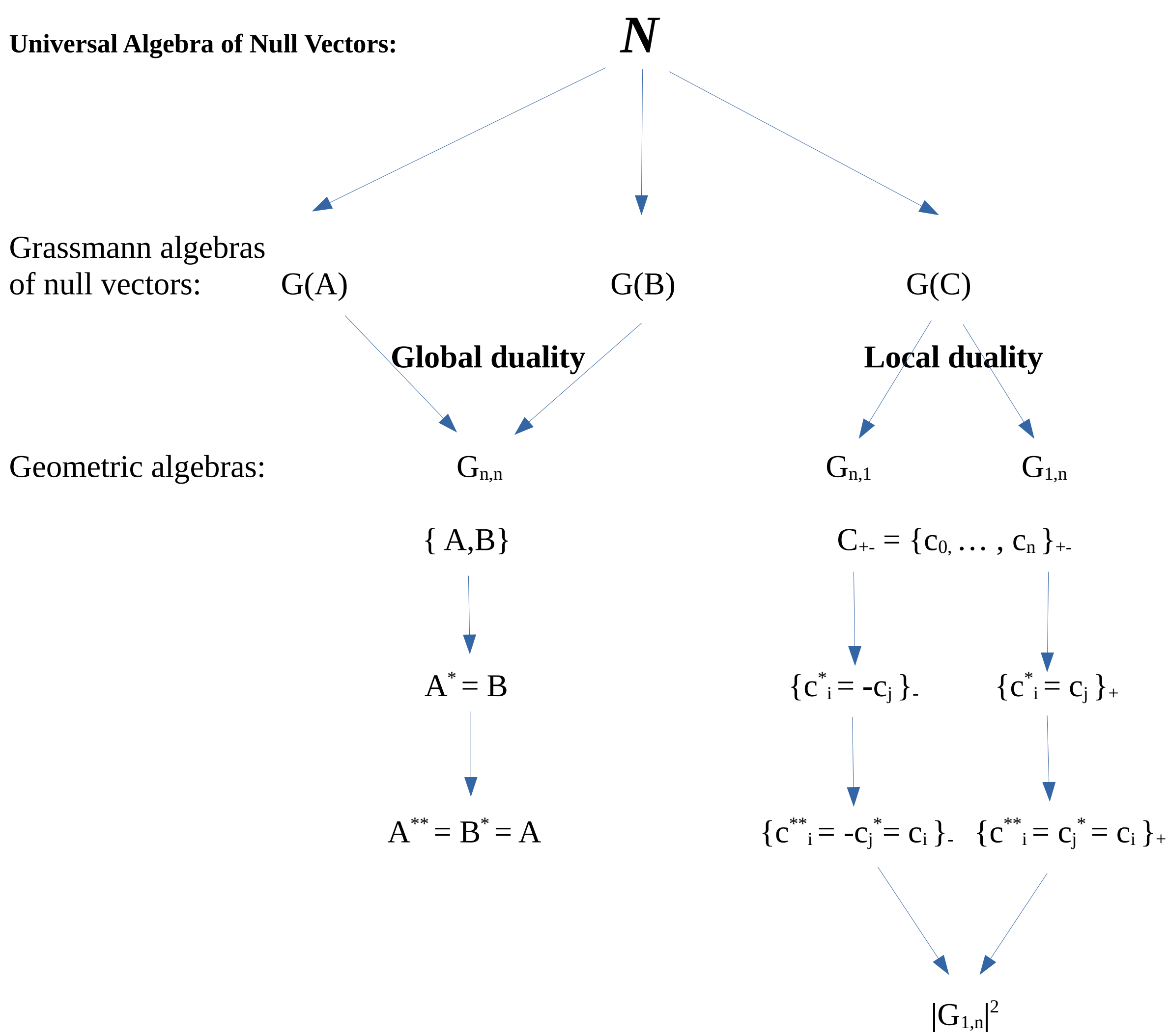
Abstract

Null vectors are *metric-free* and define *oriented rays* on the Minkowski-Einstein lightcone in *special relativity*, and in higher dimensional null cones. Separation of partial derivatives on the lightcone gives a new method for finding harmonic polynomials. Our results can be interpreted in the Nullcone Model as part of Finsler Geometry [17, 18, 19, 20, 21].

Nullvector Tree



Nullvector Tree



0. Introduction

This preprint is the most recent work of the author exploring Clifford's geometric algebras as a fundamental language for the expression of geometrical ideas across the scientific and engineering communities, and in the development of the number concept itself. For a long time I have been concerned by the criticism that the definition of a matrix, basic to all of linear algebra, should be independent of the concept of a metric. But, as is well known, the usual definition of a Clifford algebra involves the quadratic form of an inner product [3, 5, 14, 16].

The concept of a *null vector* is independent of a metric in so far as that a null vector has square 0. The concept of a vector as a *directed line segment*, can be made *metric free*, by simply defining a vector to be an *oriented ray* in a space of arbitrary dimension passing through the origin 0. Since Grassmann algebras are thought of as being metric free, defined entirely in terms of an *outer* or *exterior* product, it is natural to identify null vectors as the generating vectors of a Grassmann algebra in a larger universal algebra of null vectors, here denoted by $\mathcal{N}(\mathcal{F})$. It then follows that all higher order elements, bivectors, trivectors, ..., multivectors, characterize all the oriented directions of space independent of any metric [8]. Indeed, in this way, the concept of number itself can be defined in a metric independent way characterizing *oriented directions* [9].

Of course, these ideas became possible only after Einstein's revolutionary identification of the Minkowskian lightcone of special relativity in spacetime [4, 15]. The light cone of null vectors in the geometric algebra $\mathbb{G}_{1,n}$ makes its appearance when the generating null vector basis are assumed to have the additional structure of *local compatibility* [10]. This is in contrast to the usual, supposedly *metric free*, way of introducing into linear algebra the concept of *contravariant* and *covariant* vectors in the much larger tensor algebra. In relationship to the ideas expounded here, this is what I refer to as *global duality* in the geometric algebra $\mathbb{G}_{n,n}$. The distinction between contravariant and covarient vectors is equivalent to introducing additional structure onto a *pair* of *globally dual* Grassman algebras in the *universal algebra of null vectors* $\mathcal{N}(\mathcal{F})$, [11].

The *Universal Algebra of Null Vectors*, pictured on the previous page, constructed by adding and multiplying metric-free null vectors in Grassmann algebras, represent *oriented rays* in *abstract space*. Null vectors in $\mathcal{N}(\mathcal{F})$ representing oriented rays, are thus the basic building blocks of all the objects in linear algebra and tensor analysis, [3, 14, 9, 12, 8].

Some of the topics treated in this series of papers, either as polished publications, or on ArXiv, HAL, or as works in progress on Research Gate and YouTube as part of the *Seminar on Fundamental Problemes in Physics*, organized by Professors Jesus Cruz and William Page [19].

1 Locally Dual Grassmann algebras

Let $\mathcal{N}(\mathcal{F})$ be the **Universal Algebra of Null Vectors** [7, 10, 11].

Definition: *A null vector* $\mathbf{v} \in \mathcal{N}(\mathcal{F})$ *over the field* $\mathcal{F}$ *is an oriented direction, or ray, with the property* $\mathbf{v}^2 = 0$.

Definition: *Null vectors* $\mathbf{v} \in \mathcal{N}(\mathcal{F})$ *are added and multiplied together according to the same rules of matrix algebras over the field* $\mathcal{F}$.

Theorem: A locally dual Grassmann algebra $\mathcal{G}(\mathcal{C}_{n+1}) \subset \mathcal{N}(\mathbb{R})$ of null vectors, with $2\mathbf{c}_i \cdot \mathbf{c}_j := 1 - \delta_{ij}$ for all

$$\mathbf{c}_i, \mathbf{c}_j \in \mathcal{C}_{n+1} = \{\mathbf{c}_1, \dots, \mathbf{c}_{n+1}\},$$

and $n \geq 1$, gives the geometric algebra,

$$\mathbb{G}_{1,n} := \mathbb{R}(\mathbf{e}_1, \mathbf{f}_1, \dots, \mathbf{f}_n)$$

where, $2 \leq k \leq n$, $\mathbf{C}_{(k)} := \mathbf{c}_1 + \cdots + \mathbf{c}_k$, $\alpha_k := \frac{-\sqrt{2}}{\sqrt{k(k-1)}}$,

$$\mathbf{e}_1 := \mathbf{c}_1 + \mathbf{c}_2, \mathbf{f}_1 := \mathbf{c}_1 - \mathbf{c}_2, \mathbf{f}_k := \alpha_k \Big(\mathbf{C}_{(k)} - (k-1)\mathbf{c}_{k+1} \Big).$$

The geometric algebra $\mathbb{G}_{1,n}$ is *isomorphic* $\cong$ to the appropriate matrix algebra $\mathrm{Mat}(\mathcal{F})$, depending on Cartan-Bott periodicity $(1-n)$ modulo(8).

Example: $\mathbb{G}_{1,2} := \mathbb{R}(\mathbf{e}_1, \mathbf{f}_1, \mathbf{f}_2) = \mathbb{R}(\mathbf{e}_1, \mathbf{f}_1)(\mathbb{C}) \cong \mathbb{G}_{1,1}(\mathbb{C})$, where $j := -2\mathbf{c}_1 \wedge \mathbf{c}_2 \wedge \mathbf{c}_3$, $j^2 = -1$.

Spectral basis: $\begin{pmatrix} \mathbf{c}_2\mathbf{c}_1 \\ \mathbf{c}_1 \end{pmatrix} \begin{pmatrix} \mathbf{c}_2\mathbf{c}_1 & \mathbf{c}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{c}_2\mathbf{c}_1 & \mathbf{c}_2 \\ \mathbf{c}_1 & \mathbf{c}_1\mathbf{c}_2 \end{pmatrix}$.

Position vector: $\mathbf{x} := s_1\mathbf{e}_1 + s_2\mathbf{f}_1 + s_3\mathbf{f}_2 \in \mathbb{R}^{1,2}$, for $\mathbf{e}_1 := \mathbf{c}_1 + \mathbf{c}_2$, $\mathbf{f}_1 := \mathbf{c}_1 - \mathbf{c}_2$, and $\mathbf{f}_2 = \mathbf{c}_3 - (\mathbf{c}_1 + \mathbf{c}_2)$.

Position vector in terms of the local basis of null vectors:

$$\mathbf{x} := x_1\mathbf{c}_1 + x_2\mathbf{c}_2 + x_3\mathbf{c}_3 \in \mathbb{R}^{1,2}, \tag{1}$$

where $x_1 = s_1 + s_2 - s_3, x_2 = s_1 - s_2 - s_3, x_3 = s_3$.

Coordinate matrix $[\mathbf{g}] := \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$ for $\mathbf{g} \in \mathbb{G}_{1,2}$:

$$\mathbf{g} := \begin{pmatrix} \mathbf{c}_2\mathbf{c}_1 & \mathbf{c}_1 \end{pmatrix} [\mathbf{g}] \begin{pmatrix} \mathbf{c}_2\mathbf{c}_1 \\ \mathbf{c}_2 \end{pmatrix} = g_{11}\mathbf{c}_2\mathbf{c}_1 + g_{21}\mathbf{c}_1 + g_{12}\mathbf{c}_2 + g_{22}\mathbf{c}_1\mathbf{c}_2,$$

$$[\mathbf{c}_1] = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, [\mathbf{c}_2] = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, [\mathbf{c}_3] = \begin{pmatrix} j & 1 \\ 1 & -j \end{pmatrix},$$

where $j := -2\mathbf{c}_1 \wedge \mathbf{c}_2 \wedge \mathbf{c}_3$, $j^2 = -1$.

2 Vector gradient and Laplacian in $\mathbb{G}_{1,n}$

The *gradient*, or *vector derivative*, ∇ behaves algebraically like a vector in $\mathbb{G}_{1,n}^1$, and in addition satisfies two basic properties: 1) For any vector $\mathbf{w} \in \mathbb{G}_{1,n}^1$, $\mathbf{w} \cdot \nabla$ is the *directional derivative* at the point $\mathbf{x} \in \mathbb{R}^{1,n}$ in the direction of $\mathbf{w}$. 2) $\nabla \mathbf{x} = n+1$ counts the degrees of freedom in tangent space to the vector manifold at the point $\mathbf{x} \in \mathbb{G}_{1,n}^1$. Here, we are only considering the manifold to be $\mathbb{R}^{1,n} \equiv \mathbb{G}_{1,n}^1$ itself, [3, p.49].

The simplest case is when $n = 1$. The *position vector*

$$\mathbf{x} = s_1\mathbf{e} + s_2\mathbf{f} = x_1\mathbf{c}_1 + x_2\mathbf{c}_2 \in \mathbb{G}_{1,1}^1,$$

where $\mathbf{e} := \mathbf{c}_1 + \mathbf{c}_2$ and $\mathbf{f}_1 := \mathbf{c}_1 - \mathbf{c}_2$, which implies that

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} s_1 + s_2 \\ s_1 - s_2 \end{pmatrix} \iff \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} x_1 + x_2 \\ x_1 - x_2 \end{pmatrix}.$$

The gradient in *standard coordinates* in $\mathbb{G}_{1,1}$ is

$$\nabla := \mathbf{e}\frac{\partial}{\partial s_1} - \mathbf{f}\frac{\partial}{\partial s_2} \implies \nabla \mathbf{x} = \mathbf{e}^2 - \mathbf{f}_1^2 = 2,$$

and the *Laplacian* is

$$\nabla^2 = \frac{\partial^2}{\partial s_1^2} - \frac{\partial^2}{\partial s_2^2}.$$

In *local coordinates* in $\mathbb{G}_{1,1}$,

$$\nabla := (\nabla x_1)\frac{\partial}{\partial x_1} + (\nabla x_2)\frac{\partial}{\partial x_2} = 2\mathbf{c}_2\frac{\partial}{\partial x_1} + 2\mathbf{c}_1\frac{\partial}{\partial x_2},$$

[6, (31)]. Thus, the gradient

$$\nabla \mathbf{x} = 2(\mathbf{c}_2\mathbf{c}_1 + \mathbf{c}_1\mathbf{c}_2) = 2,$$

and the Laplacian

$$\nabla^2 = 4\,\partial_1\partial_2 \implies \nabla^2\varphi(\mathbf{x}) = 0$$

for the harmonic function $\varphi(\mathbf{x}) = (s_1^2 + s_2^2) = \frac{1}{2}(x_1^2 + x_2^2)$.

For $n = 2$, the gradient in $\mathbb{G}_{1,2}$ in the standard basis is

$$\nabla := \mathbf{e}_1\frac{\partial}{\partial s_1} - \mathbf{f}_1\frac{\partial}{\partial s_2} - \mathbf{f}_2\frac{\partial}{\partial s_3},$$

and the *Laplacian* or *Minkowski d'Alembercian* is

$$\Box \equiv \nabla^2 = \frac{\partial^2}{\partial s_1^2} - \frac{\partial^2}{\partial s_2^2} - \frac{\partial^2}{\partial s_3^2}.$$

For the position vector $\mathbf{x} := s_1\mathbf{e}_1 + s_2\mathbf{f}_1 + s_3\mathbf{f}_2$,

$$\nabla \mathbf{x} = \mathbf{e}_1^2 - \mathbf{f}_1^2 - \mathbf{f}_2^2 = 3,$$

$$\nabla \mathbf{x}^2 = 2\dot{\nabla}\dot{\mathbf{x}} \cdot \mathbf{x} = 2(\mathbf{e}_1\,\mathbf{e}_1 \cdot \mathbf{x} - \mathbf{f}_1\,\mathbf{f}_1 \cdot \mathbf{x} - \mathbf{f}_2\,\mathbf{f}_1 \cdot \mathbf{x}) = 2\mathbf{x},$$

and $\nabla^2 \mathbf{x}^2 = 2\nabla \mathbf{x} = 6$. For $\mathbf{w} \in \mathbb{G}_{1,2}^1$,

$$\mathbf{w} \cdot \nabla\,\mathbf{x} = \mathbf{w} = \nabla\,\mathbf{x} \cdot \mathbf{w}. \tag{2}$$

The *null vector gradient* in $\mathbb{G}_{1,2}$ is

$$\hat{\nabla} := \mathbf{c}_1 \frac{\partial}{\partial x_1} + \mathbf{c}_2 \frac{\partial}{\partial x_2} + \mathbf{c}_3 \frac{\partial}{\partial x_3},$$

and the *null vector Laplacian* is

$$\hat{\nabla}^2 = \partial_1\partial_2 + \partial_1\partial_3 + \partial_2\partial_3.$$

For the position vector $\mathbf{x} := x_1\mathbf{c}_1 + x_2\mathbf{c}_2 + x_3\mathbf{c}_3$,

$$\hat{\nabla}\mathbf{x} = \mathbf{c}_1^2 + \mathbf{c}_2^2 + \mathbf{c}_3^2 = 0,$$

$$\begin{aligned} \hat{\nabla}\mathbf{x}^2 &= 2\dot{\hat{\nabla}}\dot{\mathbf{x}} \cdot \mathbf{x} = 2(\mathbf{c}_1\,\mathbf{c}_1 \cdot \mathbf{x} + \mathbf{c}_2\,\mathbf{c}_2 \cdot \mathbf{x} + \mathbf{c}_3\,\mathbf{c}_3 \cdot \mathbf{x}) \\ &= \Big((x_2 + x_3)\mathbf{c}_1 + (x_1 + x_3)\mathbf{c}_2 + (x_1 + x_2)\mathbf{c}_3\Big), \end{aligned} \tag{3}$$

and the Laplacian $\hat{\nabla}^2\mathbf{x}^2 = \hat{\nabla}(\hat{\nabla}\mathbf{x}^2)$

$$= \Big(\mathbf{c}_1(\mathbf{c}_2 + \mathbf{c}_3) + \mathbf{c}_2(\mathbf{c}_1 + \mathbf{c}_3) + \mathbf{c}_3(\mathbf{c}_1 + \mathbf{c}_2)\Big) = 3. \tag{4}$$

Surprisingly $\nabla^2\mathbf{x}^2 = 6$, whereas $\hat{\nabla}^2\mathbf{x}^2 = \nabla\mathbf{x} = 3$ and $\hat{\nabla}\mathbf{x} = 0$. It is easy to generalize formulas (3) and (4) to $\mathbb{G}_{1,n}$. In $\mathbb{G}_{1,n}$, we have

$$\hat{\nabla}\mathbf{x}^2 = \sum_{i=1}^{n+1} \not{x}_i \mathbf{c}_i, \quad \text{and} \quad \hat{\nabla}^2\mathbf{x}^2 = \frac{(n+1)n}{2}, \tag{5}$$

respectively, where $\not{x}_i := (x_1 + \cdots \not{x}_i \cdots + x_{n+1})$ omitting x_i from the sum.

In [6, (31)], [8, (34)], it is shown that the vector and nullvector gradients are related by

$$\nabla = -2\hat{\nabla} + \frac{2}{n}\nabla_{(n+1)} \quad \Longleftrightarrow \quad \hat{\nabla} = -\frac{1}{2}\nabla + \frac{1}{n}\nabla_{(n+1)},$$

where the *sum gradient*

$$\nabla_{(n+1)} := \mathbf{C}_{(n+1)}\partial_{(n+1)},$$

$\mathbf{C}_{(n+1)} := \mathbf{c}_1 + \cdots + \mathbf{c}_{n+1}$ and $\partial_{(n+1)} := \partial_1 + \cdots + \partial_{n+1}$. For the Laplacian and nullvector Laplacian,

$$\nabla^2 = \left(2\hat{\nabla} + \sqrt{\frac{2(n+1)}{n}}\partial_{(n+1)}\right)\left(2\hat{\nabla} - \sqrt{\frac{2(n+1)}{n}}\partial_{(n+1)}\right),$$

[8, (36)].

Applying these formulas, for $n = 2$, the gradient in local coordinates of $\mathbb{G}_{1,2}$ is

$$\nabla = (-\mathbf{c}_1 + \mathbf{c}_2 + \mathbf{c}_3)\frac{\partial}{\partial x_1} + (\mathbf{c}_1 - \mathbf{c}_2 + \mathbf{c}_3)\frac{\partial}{\partial x_2} + (\mathbf{c}_1 + \mathbf{c}_2 - \mathbf{c}_3)\frac{\partial}{\partial x_3},$$

and the Laplacian is

$$\begin{aligned}\nabla^2 &= -(\partial_1^2 + \partial_2^2 + \partial_3^2) + \partial_1\partial_2 + \partial_1\partial_3 + \partial_2\partial_3 \\ &= (\partial_{12} - \partial_3^2) + (\partial_{13} - \partial_2^2) + (\partial_{23} - \partial_1^2). \end{aligned} \tag{6}$$

For $n = 3$, $\nabla = -2\hat{\nabla} + \frac{2}{3}\nabla_{(4)}$ and

$$\nabla^2 = 4(\hat{\nabla}^2 - \frac{1}{3}\partial_{(4)}^2) = \frac{4}{3}\left[\sum_{i<j}\partial_i\partial_j - \sum_i \partial_i^2\right].$$

For $n = 4$, $\nabla = -2\hat{\nabla} + \frac{1}{2}\nabla_{(5)}$ and

$$\nabla^2 = 4(\hat{\nabla}^2 - \frac{3}{2}\partial_{(5)}^2) = \frac{5}{2}\sum_{i<j}\partial_i\partial_j - \frac{3}{2}\sum_i \partial_i^2.$$

3 Harmonic polynomials

For $n = 2$, the Laplacian takes the form

$$\begin{aligned}\nabla^2 = \frac{\partial^2}{\partial s_1^2} - \frac{\partial^2}{\partial s_2^2} - \frac{\partial^2}{\partial s_3^2} &= \hat{\nabla}^2 - (\partial_1^2 + \partial_2^2 + \partial_3^2) \\ &= (\partial_{23} - \partial_1^2) + (\partial_{13} - \partial_2^2) + (\partial_{12} - \partial_3^2).\end{aligned}$$

By *separation of partial derivatives* in the locally dual coordinate system $\{x_1, x_2, x_3\}$ in $\mathbb{G}_{1,2}$, we can easily find a harmonic 2^{nd}-order homogeneous solution $\varphi(\mathbf{x})$. For

$$\varphi(\mathbf{x}) = (x_1^2 + 2x_2x_3) + (x_2^2 + 2x_1x_3) + (x_3^2 + 2x_1x_2),$$

$\nabla^2\varphi(\mathbf{x}) = 0$. It is not difficult to find higher order homogeneous solutions to the Laplace equation $\nabla^2\varphi(\mathbf{x}) = 0$. For example, for $\varphi(\mathbf{x}) = x_1x_2x_3 + (x_1^3 + x_2^3 + x_3^3)$ it is easily checked that $\nabla^2\varphi(\mathbf{x}) = 0$. Another 3rd order homogeneous solution found by Meta Lama 3.1 is

$$(x_1^3 + x_2^3 + x_3^3) - 3(x_1^2x_2 + x_1x_2^2 + x_2^2x_3 + x_1x_3^2 + x_2^2x_3 + x_2x_3^2).$$

The gradient in $\mathbb{G}_{1,3}$ for $n=3$ is

$$\nabla := \mathbf{e}_1\partial_{s_1} - \mathbf{f}_1\partial_{s_2} - \mathbf{f}_2\partial_{s_3} - \mathbf{f}_3\partial_{s_4} = \frac{2}{3}\nabla_{(4)} - 2\hat{\nabla},$$

where $\nabla_{(4)} = \mathbf{C}_{(4)}\partial_{(4)} = (\mathbf{c}_1 + \cdots + \mathbf{c}_4)(\partial_1 + \cdots + \partial_4)$, [6, p.9]. The position vector $\mathbf{x} \in \mathbb{G}^1_{1,3}$ is

$$\mathbf{x} = s_1\mathbf{e}_1 + s_2\mathbf{f}_1 + s_2\mathbf{f}_2 + s_4\mathbf{f}_3 = x_1\mathbf{c}_1 + x_2\mathbf{c}_2 + x_3\mathbf{c}_3 + x_4\mathbf{c}_4.$$

As usual,

$$\nabla\mathbf{x} = 4,\ \mathbf{w}\cdot\nabla\mathbf{x} = \mathbf{w} = \dot{\nabla}\,\dot{\mathbf{x}}\cdot\mathbf{w},\ \nabla\mathbf{x}^2 = 8.$$

For each coordinate x_i we have

$$\nabla x_1 = \frac{2}{3}(-2\mathbf{c}_1 + \mathbf{c}_2 + \mathbf{c}_3 + \mathbf{c}_4), \ldots, \nabla x_4 = \frac{2}{3}(\mathbf{c}_1 + \mathbf{c}_2 + \mathbf{c}_3 - 2\mathbf{c}_4),$$

and similarly, $\nabla s_1 = \mathbf{e}_1$, and $\nabla s_i = -\mathbf{f}_{i-1}$ for $i = 2,3,4$.

Recalling that $\mathbf{C}_{(k)} := \mathbf{c}_1 + \cdots + \mathbf{c}_k$. For $1 \le k \le 4$, define

$$\mathbf{D}_k := \frac{1}{2}\Big(-2\mathbf{c}_k + \mathbf{C}_{(4)}\Big),$$

and note that $\mathbf{D}_j\cdot\mathbf{D}_k = \frac{1}{2}\delta_{jk}$. It follows that $\{\mathbf{D}_1, \ldots, \mathbf{D}_4\}$ is a nullvector basis of $\mathbb{G}_{1,3}$. Other interesting properties easily follow. For example,

$$\mathbf{D}_1 + \mathbf{D}_2 = \mathbf{c}_3 + \mathbf{c}_4, \quad \mathbf{D}_1 - \mathbf{D}_2 = \mathbf{c}_2 - \mathbf{c}_1.$$

These identities should be useful in developing a representation theory based upon *local* rather than *global duality.*

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