Nullvectors, Local and Global Duality: Nullvector Manifolds and Finsler Geometry

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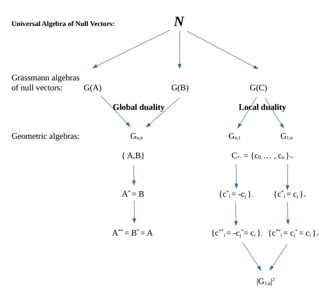
Abstract

Null vectors are *metric-free* and define *oriented rays* on the Minkowski-Einstein light-cone in *special relativity*, and in higher dimensional null cones. The concepts of *local* and *global* duality on a set of nullvectors makes possible a new classification scheme of real and complex Clifford geometric algebras. It is conjectured that Finsler Geometry is equivalent to Nullvector Manifolds.

Nullvector Tree



1. Universal Algebra of Null Vectors $\mathcal{N}(\mathcal{F})$

Definition: *A null vector* $\mathbf{v} \in \mathcal{N}(\mathcal{F})$ *over the field* $\mathcal{F}$ *is an oriented direction, or ray, with the property* $\mathbf{v}^2 = 0$.

Definition: *Null vectors* $\mathbf{v} \in \mathcal{N}(\mathcal{F})$ *are added and multiplied together according to the same rules of matrix algebras over the field* $\mathcal{F}$.

Definition: *A* 2^n*-dimensional Grassmann algebra*

$$\mathcal{G}_n(\mathcal{V}_n) \subset \mathcal{N}(\mathcal{F})$$

of multivectors is generated by a set of n *anticommuting null vectors* $\{\mathbf{v}_1, \dots, \mathbf{v}_n\} \subset \mathcal{N}^1(\mathcal{F})$, *under the operations of addition and multiplication.*

$$\mathbb{G}_{0,0,n} \equiv \mathcal{G}_n(\mathcal{V}_n) := span\{1, \mathbf{V}^{\mathbf{1}}_{\lambda_{\mathbf{1}}}, \dots, \mathbf{V}^{\mathbf{n}}_{\lambda_{\mathbf{n}}}, \}_{\mathcal{F}} \subset \mathcal{N}(\mathcal{F}),$$

where, for $1 \le k \le n$, and integers α_i,

$\mathbf{V}^k_{\lambda_k} := \{\mathbf{v}_{\alpha_1} \cdots \mathbf{v}_{\alpha_k} | \, \{1 \le \alpha_1 < \cdots < \alpha_{\binom{n}{k}} \le n\} \subset \mathbb{G}^k_{0,0,n}$

are the *oriented directions* of the $\begin{pmatrix} n \\ k \end{pmatrix}$-*basis* of null k-vector blades of in $\mathbb{G}^k_{0,0,n}$, [6] .

Example: For $\mathbf{n} = 3$, the complex Grassmann algebra of null vector blades in $\mathbb{G}_{0,0,3}(\mathbb{C})$,

$$\mathbb{G}_{0,0,3} \equiv \mathcal{G}_3(\mathbb{C}) = span\{1, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_{12}, \mathbf{v}_{13}, \mathbf{v}_{23}, \mathbf{v}_{123}\}_{\mathbb{C}},$$

for $\mathbf{v}_{ij} := \mathbf{v}_i \mathbf{v}_j \equiv \mathbf{v}_i \wedge \mathbf{v}_j = -\mathbf{v}_j \mathbf{v}_i$ and $\mathbf{v}_{123} := \mathbf{v}_1 \mathbf{v}_2 \mathbf{v}_3$.

$$|\mathbf{v}_1 + i\mathbf{v}_2|^2 := |(\mathbf{v}_1 + i\mathbf{v}_2)(\mathbf{v}_1 - i\mathbf{v}_2)| = 2|i(\mathbf{v}_1 \wedge \mathbf{v}_2)|$$

$$|(i\mathbf{v}_1 \wedge \mathbf{v}_2)(-i\mathbf{v}_1 \wedge \mathbf{v}_2)| = |\mathbf{v}_1 \mathbf{v}_2 \mathbf{v}_1 \mathbf{v}_2| = 0.$$

2. Local and Global Duality in $\mathcal{N}(\mathcal{F})$

A single null vector $\mathbf{c}_0 \in \mathcal{N}^1(\mathcal{F})$ is identified with a chosen oriented ray in any direction, and $\{\mathbf{c}_0\}_{\mathcal{F}} \widetilde{=} \mathcal{F}$.

Definition: Two null vectors $\mathbf{v_1}, \mathbf{v_2} \in \mathcal{N}^{\mathbf{1}}(\mathcal{F})$ are said to be $\pm$*conjugate* if

$$\mathbf{v}_1 \cdot \mathbf{v}_2 := \mathbf{v}_1\mathbf{v}_2 + \mathbf{v}_2\mathbf{v}_1 = \pm 1,$$

respectively. It follows that $\mathbf{v}_1\mathbf{v}_2$ and $\mathbf{v}_2\mathbf{v}_1$ are mutually annihilating idempotents that partition ± 1, respectively. Note that the conjugate null vectors $\mathbf{v}_1, \mathbf{v}_2 \in \mathcal{N}(\mathbb{C})$, are NOT anti-commutative Grassmann null vectors.

Definition: A set

$$\mathcal{V}_n = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\} \subset \mathcal{N}(\mathcal{F})$$

of null vectors are said to be *correlated* if for each $1 \le i \ne j \le n$

$$2\mathbf{v}_i \cdot \mathbf{v}_j := \mathbf{v}_i\mathbf{v}_j + \mathbf{v}_j\mathbf{v}_i \ne 0,$$

and to be *locally dual* if they are *pairwise conjugate*, i.e.

$$2\mathbf{v}_i \cdot \mathbf{v}_j := \mathbf{v}_i\mathbf{v}_j + \mathbf{v}_j\mathbf{v}_i = 1 - \delta_{ij},$$

Definition: Two Grassmann Algebras $\mathcal{G}(\mathcal{A}_n)$ and $\mathcal{G}(\mathcal{B}_n)$ in $\mathcal{N}(\mathcal{F})$, generated by the null vectors

$$\mathcal{A}_n = \{\mathbf{a}_1, \ldots \mathbf{a}_n\} \text{ and } \mathcal{B}_n = \{\mathbf{b}_1, \ldots \mathbf{b}_n\},$$

are said to be *compatible* if for all $1 \le i \le n$,

$$2\mathbf{a}_i \cdot \mathbf{b}_i := \mathbf{a}_i\mathbf{b}_i + \mathbf{b}_i\mathbf{a}_i \ne 0,$$

and *globally dual* if for all $1 \le i, j \le n$,

$$2\mathbf{a}_i \cdot \mathbf{b}_j := \mathbf{a}_i\mathbf{b}_j + \mathbf{b}_j\mathbf{a}_i = \delta_{ij},$$

[9].

3. The geometric algebras $\mathbb{G}_{n,n} \subset \mathcal{N}(\mathbb{R})$

Theorem: For $n \geq 1$, two globally dual Grassmann Algebras $\mathcal{G}(\mathcal{A}_n)$ and $\mathcal{G}(\mathcal{B}_n)$,

$$\mathcal{A}_n = \{\mathbf{a}_1, \dots \mathbf{a}_n\} \text{ and } \mathcal{B}_n = \{\mathbf{b}_1, \dots \mathbf{b}_n\},$$

define the 2^{2n}-dimensional geometric algebra

$$\mathbb{G}_{n,n}(\mathbb{R}) := \mathbb{R}(\mathbf{e}_1, \dots, \mathbf{e}_n, \mathbf{f}_1, \dots, \mathbf{f}_n),$$

where $\mathbf{e}_i := \mathbf{a}_i + \mathbf{b}_i, \mathbf{f}_i := \mathbf{a}_i - \mathbf{b}_i$ are anticommutative, and $\mathbb{G}_{n,n} \cong \mathrm{Mat}_{2^n}(\mathbb{R})$, [11, p.61].

Example: $\mathbb{G}_{1,1} := \mathbb{R}(\mathbf{e}, \mathbf{f})$

Spectral basis: $\begin{pmatrix} \mathbf{ba} \\ \mathbf{a} \end{pmatrix} \begin{pmatrix} \mathbf{ba} & \mathbf{b} \end{pmatrix} = \begin{pmatrix} \mathbf{ba} & \mathbf{b} \\ \mathbf{a} & \mathbf{ab} \end{pmatrix}.$

Position vector: $\mathbf{x} := x_1\mathbf{e} + y_1\mathbf{f} \in \mathbb{R}^{1,1}$, for $\mathbf{e} := \mathbf{a} + \mathbf{b}$,

$\mathbf{f} := \mathbf{a} - \mathbf{b}$, and satisfy $\mathbf{e}^2 = 1 = -\mathbf{f}^2$, $\mathbf{ef} = -\mathbf{fe}$.

Coordinate matrix $[\mathbf{g}] := \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$ for $\mathbf{g} \in \mathbb{G}_{\mathbf{1},\mathbf{1}}$:

$$\mathbf{g} := \begin{pmatrix} \mathbf{ba} & \mathbf{a} \end{pmatrix} [\mathbf{g}] \begin{pmatrix} \mathbf{ba} \\ \mathbf{b} \end{pmatrix} = g_{11}\mathbf{ba} + g_{21}\mathbf{a} + g_{12}\mathbf{b} + g_{22}\mathbf{ab}.$$

$$[\mathbf{a}] = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad [\mathbf{b}] = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

4. Locally Dual Grassmann algebras

Theorem: A locally dual Grassmann algebra $\mathcal{G}(\mathcal{C}_{n+1}) \subset \mathcal{N}(\mathbb{R})$ of null vectors, with $2\mathbf{c}_i \cdot \mathbf{c}_j := 1 - \delta_{ij}$ for all

$$\mathbf{c}_i, \mathbf{c}_j \in \mathcal{C}_{n+1} = \{\mathbf{c}_1, \dots, \mathbf{c}_{n+1}\},$$

and $n \geq 1$, gives the geometric algebra,

$$\mathbb{G}_{1,n} := \mathbb{R}(\mathbf{e}_1, \mathbf{f}_1, \dots, \mathbf{f}_n)$$

where, $2 \leq k \leq n$, $C_k := \mathbf{c}_1 + \cdots + \mathbf{c}_k$, $\alpha_k := \frac{-\sqrt{2}}{\sqrt{k(k-1)}}$,

$$\mathbf{e}_1 := \mathbf{c}_1 + \mathbf{c}_2, \mathbf{f}_1 := \mathbf{c}_1 - \mathbf{c}_2, \mathbf{f}_k := \alpha_k \Big(C_k - (k-1)\mathbf{c}_{k+1} \Big).$$

The geometric algebra $\mathbb{G}_{1,n}$ is *isomorphic* $\widetilde{=}$ to the appropriate matrix algebra $\mathrm{Mat}(\mathcal{F})$, depending on Cartan-Bott periodicity $(1-n)$ modulo(8).

Example: $\mathbb{G}_{1,2} := \mathbb{R}(\mathbf{e}_1, \mathbf{f}_1, \mathbf{f}_2) = \mathbb{R}(\mathbf{e}_1, \mathbf{f}_1)(\mathbb{C}) \widetilde{=} \mathbb{G}_{1,1}(\mathbb{C})$, where $j := -2\mathbf{c}_1 \wedge \mathbf{c}_2 \wedge \mathbf{c}_3$, $j^2 = -1$.

Spectral basis: $\begin{pmatrix} \mathbf{c}_2\mathbf{c}_1 \\ \mathbf{c}_1 \end{pmatrix} \begin{pmatrix} \mathbf{c}_2\mathbf{c}_1 & \mathbf{c}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{c}_2\mathbf{c}_1 & \mathbf{c}_2 \\ \mathbf{c}_1 & \mathbf{c}_1\mathbf{c}_2 \end{pmatrix}$.

Position vector: $\mathbf{x} := x_1\mathbf{e}_1 + y_1\mathbf{f}_1 + y_2\mathbf{f}_2 \in \mathbb{R}^{1,2}$, for $\mathbf{e}_1 := \mathbf{c}_1 + \mathbf{c}_2$, $\mathbf{f}_1 := \mathbf{c}_1 - \mathbf{c}_2$, and $\mathbf{f}_2 = \mathbf{c}_3 - (\mathbf{c}_1 + \mathbf{c}_2)$.

Coordinate matrix $[\mathbf{g}] := \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$ for $\mathbf{g} \in \mathbb{G}_{1,2}$:

$$\mathbf{g} := \begin{pmatrix} \mathbf{c}_2\mathbf{c}_1 & \mathbf{c}_1 \end{pmatrix} [\mathbf{g}] \begin{pmatrix} \mathbf{c}_2\mathbf{c}_1 \\ \mathbf{c}_2 \end{pmatrix} = g_{11}\mathbf{c}_2\mathbf{c}_1 + g_{21}\mathbf{c}_1 + g_{12}\mathbf{c}_2 + g_{22}\mathbf{c}_1\mathbf{c}_2,$$

$$[\mathbf{c}_1] = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, [\mathbf{c}_2] = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, [\mathbf{c}_3] = \begin{pmatrix} j & 1 \\ 1 & -j \end{pmatrix}.$$

5. The symmetric group $\mathcal{S}_{n+1}$ in $\mathbb{G}_{1,n}$

Let $\mathbb{G}_{1,n} := gen\{\mathbf{c}_1, \ldots, \mathbf{c}_{n+1}\}_{\mathbb{R}}$ in its locally dual null vector basis. In natural order let

$$\mathbf{S}_{n+1} = \mathbf{c}_1 \cdots \mathbf{c}_i \cdots \mathbf{c}_j \cdots \mathbf{c}_{n+1},$$

and let

$$(ij)\mathbf{S}_{n+1} := \mathbf{c}_1 \cdots \mathbf{c}_j \cdots \mathbf{c}_i \cdots \mathbf{c}_{n+1},$$

where only $\mathbf{c}_i$ and $\mathbf{c}_j$ are interchanged.

Theorem: The 2-cycle (ij) acting on $\mathbf{S}_n$ gives

$$(ij)\mathbf{S}_{n+1} = (-1)^{n+1}(\mathbf{c}_i - \mathbf{c}_j)\mathbf{S}_{n+1}\mathbf{c}_i - \mathbf{c}_j)$$

Proof:

In $\mathbb{G}_{1,2}$, let $\mathbf{S}_3 = \mathbf{c}_1\mathbf{c}_2\mathbf{c}_3$. Then

$$(\mathbf{c}_1 - \mathbf{c}_3)\mathbf{S}_3(\mathbf{c}_1 - \mathbf{c}_3) = -\mathbf{c}_3\mathbf{c}_1\mathbf{c}_2\mathbf{c}_3\mathbf{c}_1 = -\mathbf{c}_3(1 - \mathbf{c}_2\mathbf{c}_1)\mathbf{c}_3\mathbf{c}_1$$

$$= \mathbf{c}_3\mathbf{c}_2\mathbf{c}_1\mathbf{c}_3\mathbf{c}_1 = \mathbf{c}_3\mathbf{c}_2\mathbf{c}_1.$$

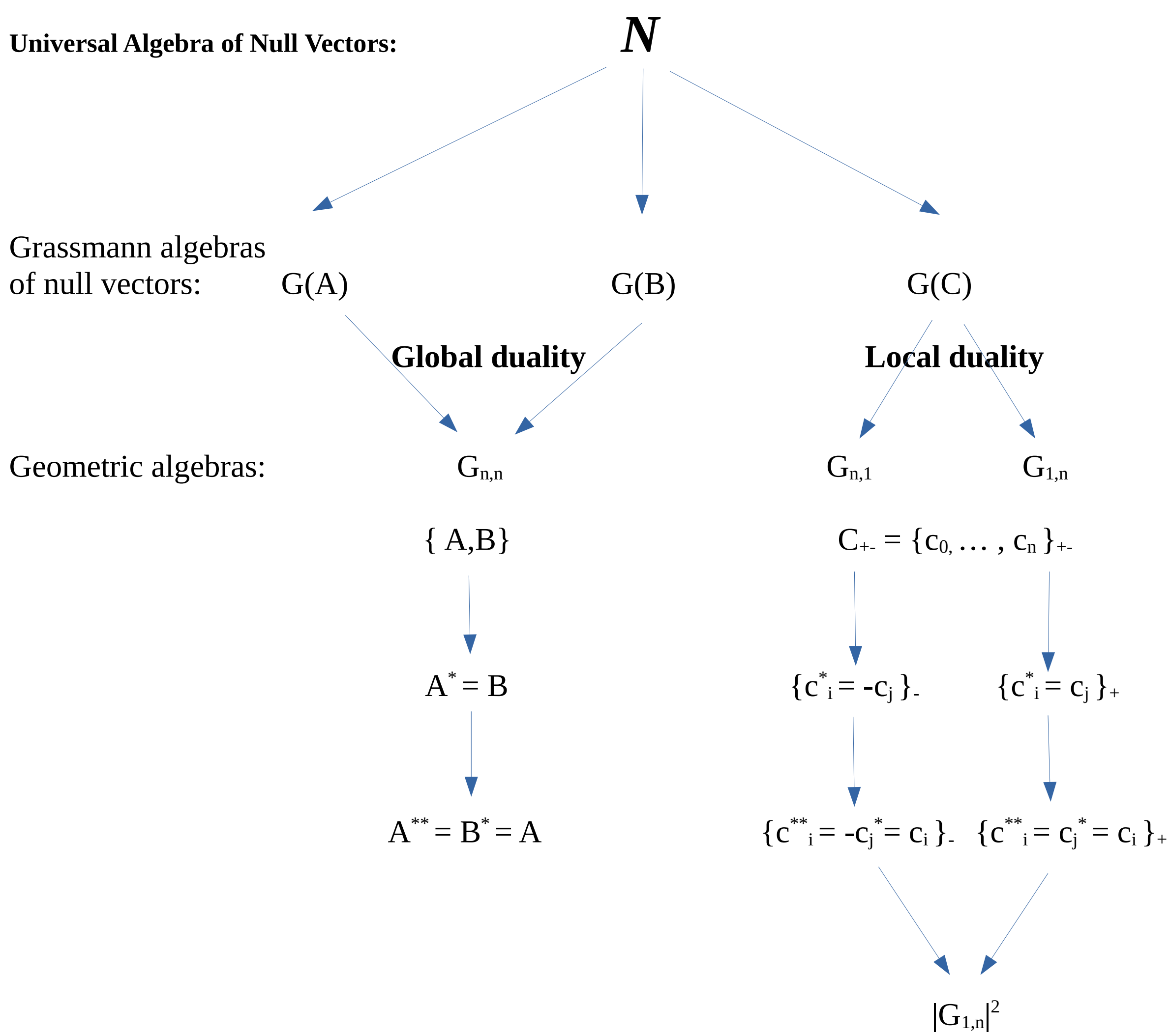
Since,

$$(\mathbf{c}_1 - \mathbf{c}_2)\mathbf{c}_3(\mathbf{c}_1 - \mathbf{c}_2) = -\mathbf{c}_3, \;\; (\mathbf{c}_1 - \mathbf{c}_2)\mathbf{c}_1(\mathbf{c}_1 - \mathbf{c}_2) = \mathbf{c}_2,$$

and $(\mathbf{c}_1 - \mathbf{c}_2)\mathbf{c}_2(\mathbf{c}_1 - \mathbf{c}_2) = \mathbf{c}_1$, the proof is easily completed. [8].

□

Nullvector Tree



6. The Fundamental Relationship between Local and Global Duality

The relationship between Clifford algebras is always confusing, none-the-less, extremely important. Starting with the concepts of a null vector and local duality, all of multilinear algebra and representation theory can be based upon these two concepts, instead of the traditional approach using (global) duality between vectors and covectors in tensor analysis and Category theory [7, 10].

The most interesting relationships between global and local duality are summarized in the following sequence of isomorphic real and complex geometric algebras $\mathbb{G}_{2n+1,0}(R)$ and $\mathbb{G}_{2n,0}(\mathbb{C})$. We have,

$$\mathbb{G}_{2n-1,1}(\mathbb{C}) \widetilde{=} \mathbb{G}_{n,n+1}(\mathbb{R}) \widetilde{=} \begin{pmatrix} \mathbb{G}_{2n,1}(\mathbb{R}) \\ \text{OR} \\ \mathbb{G}_{1,2n}(\mathbb{R}) \end{pmatrix} \widetilde{=} \mathbb{G}_{1,2n-1}(\mathbb{C}). \quad (*)$$

In the standard basis,

$$\mathbb{G}_{n,n+1}(\mathbb{R}) := gen\{\mathbf{e}_1, \ldots, \mathbf{e}_n, \mathbf{f}_1, \ldots, \mathbf{f}_n, \mathbf{f}_{n+1}\}_{\mathbb{R}}.$$

That $\mathbb{G}_{2n,0}(\mathbb{C}) \widetilde{=} \mathbb{G}_{n,n+1}(\mathbb{R})$, follows from the fact that the pseudoscalar $i := \mathbf{e}_{1\cdots n}\mathbf{f}_{1\cdots n+1}$ is in the center of $\mathbb{G}_{n,n+1}$ and $i^2 = -1$. However, there is a critical difference in the meaning of these two isomorphic, but different, geometric algebras. One of these geometric algebras is real, so that all of its multivectors have an exact geometric interpretation, whereas in the other, the meaning of *imaginary multivectors* is hypothetical. This has important theoretical implications in the interpretation of the Pauli matrices [3, 14].

Consider now the *split isomorphism*

$$\mathbb{G}_{n,n+1}(\mathbb{R}) \widetilde{=} \begin{pmatrix} \mathbb{G}_{2n,1}(\mathbb{R}) \\ \text{OR} \\ \mathbb{G}_{1,2n}(\mathbb{R}) \end{pmatrix},$$

which follows from the basic fact that either

$$\{2n-1 \text{ OR } 1-2n\} = \{3 \text{ OR } 7\} \text{ Mod } 8.$$

The split isomorphism tells us that the real geometric algebra $\mathbb{G}_{n,n+1}(\mathbb{R})$ is alternately isomorphic to $\mathbb{G}_{2n,1}(\mathbb{R})$ or $\mathbb{G}_{1,2n}(\mathbb{R})$, respectively. They are also alternately isomorphic to the corresponding matrix algebras over the complex numbers. For example,

$$\mathbb{G}_{1,2} \widetilde{=} Mat_2(\mathbb{C}), \quad \text{and} \quad \mathbb{G}_{2,1} \widetilde{=} {}^2Mat(\mathbb{R}),$$

and

$$\mathbb{G}_{1,4} \widetilde{=} {}^2Mat_2(\mathbb{H}), \quad \text{and} \quad \mathbb{G}_{4,1} \widetilde{=} Mat_4(\mathbb{C}).$$

Because signature is unimportant when considering geometric algebras over the complex numbers,

$$\mathbb{G}_{2n-1,1}(\mathbb{C}) \widetilde{=} \mathbb{G}_{2n}(\mathbb{C}) \widetilde{=} \mathbb{G}_{1,2n-1}(\mathbb{C}).$$

Every geometric algebra $\mathbb{G}_{p,q}$ has a *unique geometric structure* when it is expressed in terms of the fundamental null vectors generators of the universal null vector algebras $\mathcal{N}(\mathcal{F})$, regardless of whether in terms of real *Mother* geometric algebras $\mathbb{G}_{n,n}(\mathcal{F})$, or in terms of real *Father* geometric algebras $\mathbb{G}_{1,2n-1}(\mathcal{F})$. In both cases their corresponding isomorphic coordinate matrix representations serve as powerful computational tools.

Examples:

For $\mathbb{G}_{1,2}(\mathbb{R}) := gen\{\mathbf{e}_1, \mathbf{f}_1, \mathbf{f}_2\}_{\mathbb{R}}$, the pseudoscalar

$$i := \mathbf{e}_1\mathbf{f}_1\mathbf{f}_2 = \mathbf{e}_1(i\mathbf{f}_1)(-i\mathbf{f}_2) = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3,$$

where $\mathbf{e}_2 := i\mathbf{f}_1 = \mathbf{e}_1\mathbf{f}_2$ and $\mathbf{e}_3 := -i\mathbf{f}_2 := \mathbf{e}_1\mathbf{f}_1$. By identifying the bivectors $\mathbf{e}_1\mathbf{f}_2, \mathbf{e}_1\mathbf{f}_1 \in \mathbb{G}_{1,2}^2$ as anti-commuting basis vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 \in \mathbb{G}_{3,0}^1(\mathbb{R})$, we have almost magically turned the geometric algebra $\mathbb{G}_{1,2}$ into the isomorphic geometric algebra algebra of Euclidean space $\mathbb{G}_{3,0}$. Summarizing,

$$\mathbb{G}_{1,2} \widetilde{=} \mathbb{G}_{3,0} \widetilde{=} Mat_2(\mathbb{C}).$$

This observation is not without important physical consequence. The coordinate matrices of the Euclidean vectors makeup the Pauli matrices of quantum mechanics.

Let $j := \sqrt{-1} \in \mathbb{C}$, and $S := j\mathbf{e}_1\mathbf{f}_1\mathbf{f}_2$.

$$\mathbb{G}_{2,1}(\mathbb{R}) := \{1, \mathbf{e}_1, \mathbf{f}_1, j\mathbf{f}_2, \mathbf{e}_1\mathbf{f}_1, j\mathbf{e}_1\mathbf{f}_2, \mathbf{e}_1\mathbf{f}_2, j\mathbf{e}_1\mathbf{f}_1\mathbf{f}_2\}_{\mathbb{R}}$$

$$\mathbb{G}_{0,3}(\mathbb{R}) := \{1, j\mathbf{e}_1, \mathbf{f}_1, \mathbf{f}_2, j\mathbf{e}_1\mathbf{f}_1, j\mathbf{e}_1\mathbf{f}_2, \mathbf{f}_1\mathbf{f}_2, j\mathbf{e}_1\mathbf{f}_1\mathbf{f}_2\}_{\mathbb{R}}.$$

In each of these cases we must interpret $j \neq i = \mathbf{e}_1\mathbf{f}_1\mathbf{f}_2$, otherwise the pseudoscalar element would not be well defined.

For $\mathbb{G}_{2,1}$, note that $S_+ + S_- = 1$ and $S_+S_- = 0$. It follows that

$$\mathbb{G}_{2,1} = \{1, \mathbf{e}_1, \mathbf{f}_1, \mathbf{e}_1\mathbf{f}_1\} + J\{1, \mathbf{e}_1, \mathbf{f}_1, \mathbf{e}_1\mathbf{f}_1\} \widetilde{=} {}^2Mat(\mathbb{R}),$$

and for $\mathbb{G}_{0,3}$,

$$\mathbb{G}_{0,3} = \{1, j\mathbf{e}_1, \mathbf{f}_1, j\mathbf{e}_1\mathbf{f}_1\} + J\{1, j\mathbf{e}_1, \mathbf{f}_1, j\mathbf{e}_1\mathbf{f}_1\} \widetilde{=} {}^2\mathbb{H}.$$

Let us now consider the 5th line of the Classification Tables [4, p.217] and [11, p.74] for the geometric algebras $\mathbb{G}_{4,0}, \mathbb{G}_{3,1}, \mathbb{G}_{2,2}, \mathbb{G}_{1,3}$ and $\mathbb{G}_{0,4}$. We have

$$\mathbb{G}_{2,2} \widetilde{=} \mathbb{G}_{3,1} \widetilde{=} Mat_4(\mathbb{R}). \qquad (**)$$

The geometric algebra $\mathbb{G}_{2,2}$ is used in the Finsler Geometry Model, [16]. The geometric algebra $\mathbb{G}_{3,1}$ is used in the *conformal model* of Euclidean space [1, 12]. We have already discussed

$$\mathbb{G}_{n,n} \widetilde{=} Mat_{2n}(\mathbb{R}), \text{ and } \mathbb{G}_{n,n+1} \widetilde{=} Mat_{2n+1}(\mathbb{C}).$$

To show $\mathbb{G}_{3,1} \widetilde{=} \mathbb{G}_{2,2}$, note that

$$gen\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{f}_1\} = gen\{\mathbf{f}_1, \mathbf{e}_{123}, \mathbf{f}_1\mathbf{e}_1, \mathbf{f}_1\mathbf{e}_3\},$$

and satisfy the universal property that product of the anti commuting elements, treated as the generating vectors in $\mathbb{G}_{3,1}$, have the proper signature $\{+,+,-,-\}$, and

$$(\mathbf{f}_1)(\mathbf{e}_{123})(\mathbf{f}_1\mathbf{e}_1)(\mathbf{f}_1\mathbf{e}_3) = -\mathbf{e}_2.$$

This shows that the vector $-\mathbf{e}_1 \in \mathbb{G}_{3,1}^1$ is the pseudoscalar element $\mathbf{f}_1\mathbf{e}_{123}\mathbf{f}_1\mathbf{e}_1\mathbf{f}_1\mathbf{e}_3 \in \mathbb{G}_{2,2}^4$.

We now identify the isomorphisms

$$\mathbb{G}_{1,3} \widetilde{=} \mathbb{G}_{0,4} \widetilde{=} \mathbb{G}_{4,0} \widetilde{=} Mat_2(\mathbb{H}).$$

For $\mathbb{G}_{1,3}$, let $\mathbf{S} := \mathbf{f}_{123}, \mathbf{S}_\pm = \frac{1}{2}(1 \pm \mathbf{S})$. Then

$$\mathbb{G}_{1,3} = span\{1, \mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_{12}\} + span\mathbf{S}\{1, \mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_{12}\} \widetilde{=} Mat_2(\mathbb{H}).$$

The locally dual geometric algebra $\mathbb{G}_{1,3}$, see equation (*), models the famous Dirac matrices, and is the *Spacetime Algebra* of special relativity [3]. For $\mathbb{G}_{0,4}$, let $\mathbf{S} := \mathbf{f}_{1234}, \mathbf{S}_\pm = \frac{1}{2}(1 \pm \mathbf{S})$. Then

$$\mathbb{G}_{0,4} = span\{1, \mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_{12}\} + span\mathbf{S}\{1, \mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_{12}\} \widetilde{=} Mat_2(\mathbb{H}),$$

and finally for $\mathbb{G}_{4,0}$

$$\mathbb{G}_{4,0} \widetilde{=} gen\{\mathbf{e}_1, \mathbf{f}_1\mathbf{e}_1, \mathbf{f}_2\mathbf{e}_1, \mathbf{f}_3\mathbf{e}_1\} \widetilde{=} \mathbb{G}_{1,3} \widetilde{=} Mat_2(\mathbb{H}).$$

It is possible that each of these isomorphisms might have different physical consequences [1, 3, 12, 5].

7. Nullvector Manifolds and Finsler geometry

In his 1996 paper, "Finsler Geometry is Just Riemannian Geometry without the Quadratic Restriction", Shiing-Shen Chern, gives an introduction to Finsler Geometry, and with coauthors, wrote a comprehensive book on the subject [16, 18]. Other research has explored the topic in terms of Kaluza-Klein structures [17]. In a just announced Ph.D. thesis, Sjors Heefer examines the possibility that Finsler Geometry may be the key to the long sought after unification of quantum mechanics with general relativity [19].

In [2, 15, 13], the authors explored the concept of a *vector manifold* as a coordinate free approach to manifolds, usually expressed in terms of differential forms and tensor analysis. We generalize the concept of a vector manifold to *two kinds* of Finsler Geometries (FM), a *globally dual* FM and a *locally dual* FM in the appropriately defined global and local *Nullvector Manifolds*.

In a vector manifold, *partial derivatives* playing the role of contravariant vectors, are replaced by *directional derivatives*, and covariant vectors are replaced by their *dual* nullvector equivalents. In *global duality*, we replace contravariant and covariant vectors by nullvectors $\mathbf{a}_i$ and

$\mathbf{b}_j$,

$$\frac{\partial}{\partial x^i} \to \mathbf{a}_i = \mathbf{a}_i \cdot \nabla_{\mathbf{F}} \, \mathbf{x} \quad \text{and} \quad \frac{\partial \mathbf{y}}{\partial y^j} \to \mathbf{b}_j = \nabla_{\mathbf{F}} \varphi_j(\mathbf{y}),$$

respectively. In addition,

$$\mathbf{a}_i \cdot \mathbf{b}_j := \frac{1}{2}(\mathbf{a}_i \mathbf{b}_i + \mathbf{b}_i \mathbf{a}_i) = \frac{1}{2}\delta_{ij}$$

guarantees that $\mathbf{a}_i, \mathbf{b}_j \in \mathbb{G}^1_{n,n} \widetilde{=} Mat_{2^n}(\mathbb{R})$

Local duality is a different story. A set of $(n+1)$-nullvectors $\{\mathbf{c}_0, \ldots, \mathbf{c}_n\}$ are said to be *locally dual* if each pair of nullvectors $\mathbf{c}_i, \mathbf{c}_j$ satisfies

$$\frac{\partial}{\partial x^i} \to \mathbf{c}_i := \mathbf{c}_i \cdot \nabla_{\mathbf{F}} \, \mathbf{x} \quad \text{and} \quad \frac{\partial \mathbf{x}}{\partial x^j} \to \mathbf{c}_j := \nabla_{\mathbf{F}} \, \varphi(\mathbf{x}),$$

where $\nabla_{\mathbf{F}}$ is the *nullvector gradient.* In addition,

$$\mathbf{c}_i \cdot \mathbf{c}_j := \frac{1}{2}(\mathbf{c}_i \mathbf{c}_i + \mathbf{c}_i \mathbf{c}_i) = \frac{1}{2}(1 - \delta_{ij}),$$

guarantees that $\mathbf{c}_i, \mathbf{c}_j \in \mathbb{G}^1_{1,n}$. In spaces that are locally dual, **and** $n+1$ is **even**, the distinction between *contravariance* and *covariance* is lost, or both are contravariant and covariant at the same time! In the case $n = 3$, the two different geometric algebras $\mathbb{G}_{2,2}(\mathbb{R}) \widetilde{=} \mathbb{G}_{3,1}(\mathbb{R})$ are isomorphic to the same matrix algebra $Mat_4(\mathbb{R})$, see equation (**).

We now show how S-S. Chern's Finsler Geometry translates into equivalent globally and locally dual geometric Nullvector Manifolds. A *locally dual* Finsler Geometry is a pair $(F(\mathbf{x}, \mathbf{y}), \mathbf{M})$ defined by $n+1$ points on the nullcone in the geometric algebra $\mathbb{G}_{1,n}$, where both the tangent and cotangent bundles at the point $\mathbf{x} \in \mathbf{M}$ is spanned by

$$\mathbf{TF} = \mathbf{T}^*\mathbf{M} \equiv span\{\mathbf{c}_0, \mathbf{c}_1, \ldots, \mathbf{c}_n\} = \frac{1}{2}\nabla_{\mathbf{F}}[\mathbf{F}^2] = \mathbb{G}^1_{1,n}.$$

In this case, the metric is defined by the $(\mathbf{x}, \mathbf{y})$-Hessians

$$[g_{ij}] := 2\mathbf{c}_i \cdot \mathbf{c}_j = \frac{1}{2}\{[\mathbf{F}^2_{y_i y_j}]\} = \frac{1}{2}\{[\mathbf{F}^2_{x_i x_j}]\} = [1 - \delta_{ij}],$$

[16], and [18, pp.5-8].

A *globally dual* Finsler Geometry is a pair $(F(\mathbf{x}, \mathbf{y}), \mathbf{M})$ defined by $2n$ points on the nullcone in the geometric algebra $\mathbb{G}_{n,n}$, where the tangent bundle $\mathbf{TM}$ and the cotangent bundle $\mathbf{T}^*\mathbf{M}$ are spanned by the sets $\{\mathbf{a}_1, \cdots, \mathbf{a}_n\}$ and $\{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$, respectively. That is

$$\mathbf{TM} \equiv span\{\mathbf{a}_1, \ldots, \mathbf{a}_n\} \cdot \nabla_{\mathbf{F}}\, \mathbf{x} \subset \mathbb{G}_{n,n},$$

and

$$\mathbf{T}^*\mathbf{M} \equiv \nabla_{\mathbf{F}}\, \{\varphi(\mathbf{y})\} = span\{\mathbf{b}_1, \ldots, \mathbf{b}_n\} \subset \mathbb{G}^1_{n,n},$$

In this case, the metric is defined by the $(\mathbf{x}, \mathbf{y})$-Hessians

$$[g_{ij}] := 2\mathbf{a}_i \cdot \mathbf{b}_j = \frac{1}{2}\{[\mathbf{F}^2_{x_i y_j}]\} = \frac{1}{2}\{[\mathbf{F}^2_{y_i x_j}]\} = [\delta_{ij}].$$

Finally, as might be expected, a *complex Finsler Geometry* is defined by complexifiying both the global and local nullvector manifolds over the complex numbers $\mathbb{C}$. See equation $(*)$.

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issues raised by the thought provoking participants regarding my conjectured equivalence between the Finsler and Nullvector Geometry Models. I am very grateful to the Organizers, Jesus Cruz Guzman and William Page for inviting me to give this presentation [20].

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