The Universal Null Substrate:

Emergence of Spacetime, Observer Bridges, and Quantum Algebras

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Abstract

We present a purely algebraic, matrix-free foundation for spacetime, conformal, and quantum geometric algebras emerging from a Universal Null Substrate (UNS). Starting from primitive null vectors governed solely by $n^2 = 0$ and linearity, the inner product $n_1 \cdot n_2 := \frac{1}{2}(n_1 n_2 + n_2 n_1)$ emerges directly from expanding the square $(n_1 + n_2)^2$. Introducing dual four-dimensional null substrates V_4^c and V_4^w with $c_i \cdot c_j = \frac{1}{2}(1 - \delta_{ij})$ and $w_i \cdot w_j = -\frac{1}{2}(1 - \delta_{ij})$ defines the Quadratic Space of Null Vectors (QSNV), yielding the orthonormal symmetric Hadamard bases $\{e_1, f_1, f_2, f_3\}$ of the Spacetime algebra $\mathbb{G}(1,3)$ and $\{\tilde{f}_1, \tilde{e}_1, \tilde{e}_2, \tilde{e}_3\}$ of the Majorana algebra $\mathbb{G}(3,1)$. Furthermore, we show how extending the Pauli-equivalent algebra $\mathbb{G}(1,2) \to \mathbb{G}(1,2,1)$ via a single substrate generator provides a metric-free nilpotent translational bridge for relativistic observers. Crucially, factoring this shifted observer idempotent as a single null dyad $\kappa_1' = \frac{1}{2} c_1 w_1'$ absorbs nilpotent Jordan shears directly into the lightcone without forming normal products $A^T A$, drastically simplifying classical Singular Value Decomposition (SVD) and complementing recent matrix-free Clifford SVD methods. Bridging V_4^c and V_4^w via $c_i \cdot w_j = \delta_{ij}$ embeds the Conformal Spacetime algebras $\mathbb{G}(2,4)$ and $\mathbb{G}(4,2)$ inside the full phase space $\mathbb{G}(4,4) = \mathbb{G}(1,3) \oplus^{\perp} \mathbb{G}(3,1)$, naturally accounting for the conformal group $O(2,4)$, Dirac's six-cone, Penrose twistors, and the four mutually annihilating composite idempotents $\Phi_1, \ldots, \Phi_4$ with invariant scalar floor $\langle \Phi_a \rangle_0 = \frac{1}{4}$ without recourse to matrix representations.

Contents

1 Part I: The Pure Algebraic Substrate **2**
- 1.1 Foundational Number Systems (Item 1) 2
- 1.2 The Primitive Null Vector and Linearity (Items 2–3) 2
- 1.3 Expanding the Square and the Emergent Metric (Items 4–5) 3

2 Part II: Dual Bases of the Spacetime and Majorana Algebras **3**
- 2.1 Substrate Interactions and Geometric Product Rules (Items 6–7) 3
- 2.2 The Quadratic Space of Null Vectors and Hadamard Bases (Items 8–9) . . . 3

3 Part III: Conformal Bridge, Observer Splitting, and Phase Space **4**
3.1 The Conformal Bridge Between V_4^c and V_4^w (Item 10) 4
3.2 Conformal Algebras and the 8D Phase Space $\mathbb{G}(4,4)$ (Item 11) 5
3.3 Quantum Algebras and the $\mathbb{G}(1,2) \to \mathbb{G}(1,2,1)$ Observer Bridge (Item 12) . 5
3.4 Null Factorization and the Simplification of Singular Value Decomposition (SVD) 7

4 Part IV: Physical Meaning: The Conformal Group $O(2,4)$ and Twistors **8**
4.1 The Conformal Group of Spacetime (Item 13) 8
4.2 Dirac's Six-Cone from the Null Substrate (Item 14) 9
4.3 Penrose Twistors and Zero Residue Factorization (Item 15) 9

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1 Part I: The Pure Algebraic Substrate

Traditional formulations of geometric algebra and mathematical physics begin by postulating a real vector space $\mathbb{R}^{p,q}$ equipped with an *a priori* non-degenerate quadratic form of signature (p,q). In the Universal Null Substrate (UNS) framework, we invert this hierarchy: the metric is not assumed at the outset, but emerges algebraically from the associative multiplication of primitive null generators.

1.1 Foundational Number Systems (Item 1)

The elementary building blocks of geometric algebra reside within low-dimensional real Clifford algebras:

- **Hyperbolic Numbers $\mathbb{G}(1,0)$:** Generated by a single unit element e_1 with $e_1^2 = +1$. It possesses the canonical spectral basis of primitive idempotents $u_\pm = \frac{1}{2}(1 \pm e_1)$ satisfying $u_+ + u_- = 1$, $u_\pm^2 = u_\pm$, and $u_+ u_- = 0$.
- **Complex Numbers $\mathbb{G}(0,1)$:** Generated by a single anti-Euclidean unit f_1 with $f_1^2 = -1$.
- **Quaternions $\mathbb{G}(0,2)$:** Generated by two anticommuting imaginary units f_1, f_2 with $f_1^2 = f_2^2 = -1$ and $(f_1 f_2)^2 = -1$.

Rather than treating $\mathbb{G}(1,0)$, $\mathbb{G}(0,1)$, and $\mathbb{G}(0,2)$ as unrelated axioms, the Universal Null Substrate reveals them as different interference patterns of primitive null elements.

1.2 The Primitive Null Vector and Linearity (Items 2–3)

Definition 1.1 (Primitive Null Vector)**.** *A primitive null vector n represents a pure oriented direction of zero length, governed by the single algebraic axiom:*

$$n^2 = 0. \tag{1}$$

We assume that a collection of distinct primitive null vectors $\{n_1, n_2, \ldots, n_m\}$ spans a standard linear vector space V over the real numbers $\mathbb{R}$, so that any linear combination $\alpha_1 n_1 + \alpha_2 n_2$ ($\alpha_i \in \mathbb{R}$) obeys standard distributivity and associativity under the geometric product.

1.3 Expanding the Square and the Emergent Metric (Items 4–5)

Consider the sum of two distinct null vectors $n_1, n_2 \in V$. Although each generator is individually nilpotent ($n_1^2 = n_2^2 = 0$), the square of their sum is governed by their cross-interaction:

$$(n_1 + n_2)^2 = n_1^2 + n_1 n_2 + n_2 n_1 + n_2^2 = n_1 n_2 + n_2 n_1. \tag{2}$$

Because $(n_1 + n_2)^2$ is symmetric under the exchange $n_1 \leftrightarrow n_2$, we require that the symmetrized product takes values in the ground field $\mathbb{R}$. This allows us to *define* the emergent bilinear inner product directly from the symmetric part of the geometric product:

$$n_1 \cdot n_2 := \frac{1}{2}(n_1 n_2 + n_2 n_1) \in \mathbb{R}, \qquad n_1 \wedge n_2 := \frac{1}{2}(n_1 n_2 - n_2 n_1). \tag{3}$$

Consequently, the fundamental decomposition $n_1 n_2 = n_1 \cdot n_2 + n_1 \wedge n_2$ is not an external metric postulate; it is the direct algebraic consequence of evaluating $(n_1 + n_2)^2$ in an associative ring generated by null elements.

2 Part II: Dual Bases of the Spacetime and Majorana Algebras

To construct four-dimensional relativistic physics without matrices, we introduce two complementary 4-dimensional null substrates: the **cohesive substrate** $V_4^c = \mathrm{span}_{\mathbb{R}}\{c_1, c_2, c_3, c_4\}$ and the **repulsive (dual) substrate** $V_4^w = \mathrm{span}_{\mathbb{R}}\{w_1, w_2, w_3, w_4\}$.

2.1 Substrate Interactions and Geometric Product Rules (Items 6–7)

Every basis generator in V_4^c and V_4^w is strictly nilpotent:

$$c_i^2 = 0, \qquad w_i^2 = 0 \qquad \forall\, i \in \{1, 2, 3, 4\}. \tag{4}$$

The internal cohesion of V_4^c and mutual repulsion of V_4^w are defined by uniform constant off-diagonal inner products for all $i, j \in \{1, 2, 3, 4\}$:

$$c_i \cdot c_j := \frac{1}{2}(1 - \delta_{ij}), \qquad w_i \cdot w_j := -\frac{1}{2}(1 - \delta_{ij}). \tag{5}$$

Using $2(x \cdot y) = xy + yx$, Equation (5) immediately yields the fundamental non-commutative geometric product rules for $i \neq j$:

$$\boxed{c_i c_j = 1 - c_j c_i = \frac{1}{2} + c_i \wedge c_j} \quad \text{and} \quad \boxed{w_i w_j = -1 - w_j w_i = -\frac{1}{2} + w_i \wedge w_j}. \tag{6}$$

2.2 The Quadratic Space of Null Vectors and Hadamard Bases (Items 8–9)

Theorem 2.1 (Emergence of $\mathbb{G}(1,3)$ and $\mathbb{G}(3,1)$ via Symmetric Hadamard Combinations)**.** *1. Forming symmetric Hadamard combinations of the cohesive basis $\{c_1, c_2, c_3, c_4\} \subset V_4^c$*

produces a standard orthonormal basis $\{e_1, f_1, f_2, f_3\}$ *of Hestenes'* ***Spacetime Algebra (Dirac Algebra)*** $\mathbb{G}(1,3) := \mathrm{gen}(V_4^c)$*:*

$$\begin{aligned} e_1 &= \frac{1}{\sqrt{6}}\sum_{i=1}^{4} c_i = \frac{1}{\sqrt{6}}(c_1 + c_2 + c_3 + c_4), \qquad & e_1^2 &= +1, \\ f_1 &= \frac{1}{\sqrt{2}}(c_1 - c_2 - c_3 + c_4), & f_1^2 &= -1, \\ f_2 &= \frac{1}{\sqrt{2}}(c_1 + c_2 - c_3 - c_4), & f_2^2 &= -1, \\ f_3 &= \frac{1}{\sqrt{2}}(c_1 - c_2 + c_3 - c_4), & f_3^2 &= -1. \end{aligned} \tag{7}$$

2. *Dually, replacing the cohesive primitives* c_i *with the repulsive primitives* $\{w_1, w_2, w_3, w_4\} \subset V_4^w$ *produces a standard orthonormal basis* $\{\tilde{f}_1, \tilde{e}_1, \tilde{e}_2, \tilde{e}_3\}$ *of the* ***Majorana (Anti-Spacetime) Algebra*** $\mathbb{G}(3,1) := \mathrm{gen}(V_4^w)$*:*

$$\begin{aligned} \tilde{f}_1 &= \frac{1}{\sqrt{6}}\sum_{i=1}^{4} w_i = \frac{1}{\sqrt{6}}(w_1 + w_2 + w_3 + w_4), \qquad & \tilde{f}_1^2 &= -1, \\ \tilde{e}_1 &= \frac{1}{\sqrt{2}}(w_1 - w_2 - w_3 + w_4), & \tilde{e}_1^2 &= +1, \\ \tilde{e}_2 &= \frac{1}{\sqrt{2}}(w_1 + w_2 - w_3 - w_4), & \tilde{e}_2^2 &= +1, \\ \tilde{e}_3 &= \frac{1}{\sqrt{2}}(w_1 - w_2 + w_3 - w_4), & \tilde{e}_3^2 &= +1. \end{aligned} \tag{8}$$

Proof. By direct evaluation of the symmetric null index arithmetic, since $c_i^2 = 0$ and $\sum_{i\neq j} c_i \cdot c_j = \frac{1}{2}(12) = 6$, we obtain $e_1^2 = \frac{6}{6} = +1$. Similarly, for each balanced difference combination, $(c_1 \pm c_2 \mp c_3 \mp c_4)^2 = -2$, so dividing by $\sqrt{2}$ yields $f_k^2 = -1$ $(k = 1, 2, 3)$. Furthermore, because $(c_i - c_j)\cdot(c_k - c_l) = 0$ for disjoint index pairs, these vectors are mutually orthogonal ($e_1 f_k = -f_k e_1$ and $f_j f_k = -f_k f_j$ for $j \neq k$). For V_4^w, the repulsive inner product $w_i \cdot w_j = -\frac{1}{2}$ $(i \neq j)$ naturally inverts the signs of all quadratic norms, yielding $\tilde{f}_1^2 = -1$ and $\tilde{e}_1^2 = \tilde{e}_2^2 = \tilde{e}_3^2 = +1$. □

3 Part III: Conformal Bridge, Observer Splitting, and Phase Space

3.1 The Conformal Bridge Between V_4^c and V_4^w (Item 10)

Taken separately, V_4^c generates $\mathbb{G}(1,3)$ and V_4^w generates $\mathbb{G}(3,1)$. Together, we form the 8-dimensional **Quadratic Space of Null Vectors (QSNV)** $V^{cw} = V_4^c \oplus V_4^w$ by defining the cross-duality bridge condition:

$$c_i \cdot w_j := \delta_{ij} \iff c_i w_j + w_j c_i = 2\delta_{ij} \iff c_i w_j = \delta_{ij} - w_j c_i = \delta_{ij} + c_i \wedge w_j, \tag{9}$$

for $i, j \in \{1, 2, 3, 4\}$. Notice in particular that for distinct indices $i \neq j$, cohesive and repulsive generators strictly anticommute: $c_i w_j = -w_j c_i$.

We express the transformation from each mutually dual null basis pair (c_k, w_k) to the symmetric and antisymmetric orthonormal bridge generators P_k and N_k via the normalized

2×2 Hadamard matrix:

$$\begin{bmatrix} P_k \\ N_k \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} c_k \\ w_k \end{bmatrix} \iff P_k := \frac{1}{\sqrt{2}}(c_k + w_k), \qquad N_k := \frac{1}{\sqrt{2}}(c_k - w_k). \tag{10}$$

Proposition 3.1 (Properties of the Bridge Generators and Microscopic Entanglement)**.** *For each $k \in \{1, 2, 3, 4\}$, the vectors P_k and N_k are mutually orthogonal unit vectors of opposite signature:*

$$P_k^2 = \frac{1}{2}(c_k^2 + w_k^2 + 2c_k \cdot w_k) = +1, \qquad N_k^2 = \frac{1}{2}(c_k^2 + w_k^2 - 2c_k \cdot w_k) = -1, \tag{11}$$

and $P_k \cdot N_k = \frac{1}{2}(c_k^2 - w_k^2) = 0$. However, for distinct indices $i \neq j$, they exhibit a mandatory off-diagonal coupling inherited from the substrates:

$$P_i \cdot N_j = \frac{1}{2}(c_i + w_i) \cdot (c_j - w_j) = \frac{1}{2}\left(c_i \cdot c_j - c_i \cdot w_j + w_i \cdot c_j - w_i \cdot w_j\right) = \frac{1}{2}\left(\frac{1}{2} - 0 + 0 - \left(-\frac{1}{2}\right)\right) = \frac{1}{2}, \tag{12}$$

while $P_i \cdot P_j = 0$ and $N_i \cdot N_j = 0$ for all $i \neq j$.

Remark 3.2 (Microscopic Entanglement and Orthogonal Separation)**.** *The forced geometric overlap $P_i \cdot N_j = \frac{1}{2}$ $(i \neq j)$ proves that kinematic properties cannot be naively isolated node-by-node within the substrate: while $\{P_k\}$ spans $\mathbb{R}^{4,0}$ and $\{N_k\}$ spans $\mathbb{R}^{0,4}$, the two subspaces are tilted relative to each other. However, by using the symmetric Hadamard combinations of Equations* (7)–(8) *or pairing disjoint index planes, orthogonal hyperbolic planes $\mathbb{G}(1, 1)$ and commuting projectors can be extracted cleanly.*

3.2 Conformal Algebras and the 8D Phase Space $\mathbb{G}(4, 4)$ (Item 11)

Each matched pair $\{c_k, w_k\}$ with $c_k \cdot w_k = 1$ generates a hyperbolic plane $\mathbb{G}(1, 1) = \text{gen}\{P_k, N_k\}$ containing a pair of complementary primitive idempotents:

$$\kappa_k := \frac{1}{2}c_k w_k, \qquad \bar{\kappa}_k := \frac{1}{2}w_k c_k = 1 - \kappa_k, \qquad \kappa_k^2 = \kappa_k, \qquad \kappa_k \bar{\kappa}_k = 0. \tag{13}$$

By orthogonally adjoining one or more hyperbolic planes $\mathbb{G}(1, 1)$ across the bridge between $\mathbb{G}(1, 3)$ and $\mathbb{G}(3, 1)$, we obtain the hierarchy of physical phase spaces:

$$\mathbb{G}(2, 4) = \mathbb{G}(1, 3) \oplus^{\perp} \mathbb{G}(1, 1) \qquad \text{(Conformal Spacetime Algebra)}, \tag{14}$$

$$\mathbb{G}(4, 2) = \mathbb{G}(3, 1) \oplus^{\perp} \mathbb{G}(1, 1) \qquad \text{(Conformal Majorana Algebra)}, \tag{15}$$

$$\mathbb{G}(4, 4) = \text{gen}(V_4^c \oplus V_4^w) \cong \mathbb{G}(1, 3) \oplus^{\perp} \mathbb{G}(3, 1) \qquad \text{(Full 8D Phase Space)}. \tag{16}$$

3.3 Quantum Algebras and the $\mathbb{G}(1, 2) \to \mathbb{G}(1, 2, 1)$ Observer Bridge (Item 12)

Restricting to the first three cohesive generators $\{c_1, c_2, c_3\} \subset V_4^c$, we form the 3-dimensional subalgebra $\mathbb{G}(1, 2)$ via the combinations:

$$\mathbf{e}_0 = \frac{1}{\sqrt{3}}(c_1 + c_2 + c_3) \ \ (\mathbf{e}_0^2 = +1), \quad \mathbf{f}_1 = c_1 - c_2 \ \ (\mathbf{f}_1^2 = -1), \quad \mathbf{f}_2 = \frac{1}{\sqrt{3}}(c_1 + c_2 - 2c_3) \ \ (\mathbf{f}_2^2 = -1). \tag{17}$$

As is fundamental in geometric algebra, the even subalgebras of both Spacetime and Majorana algebras are isomorphic to the Pauli Algebra $\mathbb{G}(3, 0)$ of quantum mechanics, and directly at the full algebra level:

$$\mathbb{G}^+(1, 3) \cong \mathbb{G}(3, 0) \cong \mathbb{G}^+(3, 1), \qquad \text{and} \qquad \mathbb{G}(1, 2) \cong \mathbb{G}(3, 0) \text{ (over } \mathbb{C} \text{ or via dual split)}. \tag{18}$$

The $\mathbb{G}(1,2) \to \mathbb{G}(1,2,1)$ Translational Bridge and Oziewicz Observers: Suppose an observer in $\mathbb{G}(1,2)$ is characterized by the idempotent $\kappa_1 = \frac{1}{2}c_1 w_1$ ($\kappa_1^2 = \kappa_1$). In Zbigniew Oziewicz's metric-free theory of relativity (developed further with J. J. Cruz Guzmán and Bill Page), an observer is defined as a $(1,1)$-tensor idempotent $\kappa = \tau \otimes T$ with $\tau(T) = 1$. In our QSNV framework, $\kappa_1 = \frac{1}{2}c_1 w_1$ realizes this exact idempotent without category theory: $c_1 \in V_4^c$ acts as the contravariant Pfaff time 1-form τ, and $w_1 \in V_4^w$ acts as the covariant state vector T.

Now, before closing the full metric $\mathbb{G}(1,3)$, suppose we adjoin a fourth null element $\mathbf{n}_0$ (constructed in the orthogonal complement of $\text{span}\{c_1, c_2, c_3, w_1\}$ inside the bridge) that is completely degenerate with respect to $\mathbb{G}(1,2)$: $\mathbf{n}_0^2 = 0$ and $\mathbf{n}_0 \cdot x = 0$ for all $x \in \text{span}\{c_1, c_2, c_3, w_1\}$. This forms the degenerate Clifford algebra $\mathbb{G}(1,2,1)$. Shifting $w_1 \mapsto w_1' = w_1 + 2\beta\mathbf{n}_0$ transforms the observer idempotent to:

$$\kappa_1' = \frac{1}{2}c_1(w_1 + 2\beta\mathbf{n}_0) = \kappa_1 + \beta c_1\mathbf{n}_0, \qquad (\kappa_1')^2 = \kappa_1'. \tag{19}$$

Because $(\beta c_1\mathbf{n}_0)^2 = 0$, this transition is a **pure nilpotent Galilean/null translation** across the bridge $\mathbb{G}(1,2) \to \mathbb{G}(1,2,1)$ that preserves the clock 1-form c_1 globally and introduces no Wigner rotation (Thomas precession).

Concrete Verification in the QSNV Basis: Using only the substrate rules $c_i c_j + c_j c_i = 1$ $(i \neq j)$, $w_i w_j + w_j w_i = -1$ $(i \neq j)$, and $c_i w_j + w_j c_i = 2\delta_{ij}$, we verify these exact algebraic properties:

1. **Step-by-Step Verification of Idempotency $(\kappa_1')^2 = \kappa_1'$:** Let $\kappa_1 = \frac{1}{2}c_1 w_1$, and let $\mathbf{n}_0$ denote the radical generator of $\mathbb{G}(1,2,1)$ satisfying $\mathbf{n}_0^2 = 0$ and $\mathbf{n}_0 x = -x\mathbf{n}_0$ for all $x \in \text{span}\{c_1, c_2, c_3, w_1\}$. Define the nilpotent transition term $\Delta := \beta c_1\mathbf{n}_0$ ($\beta \in \mathbb{R}$), so that $\kappa_1' = \kappa_1 + \Delta$. Using $c_1^2 = 0$, $\mathbf{n}_0^2 = 0$, $c_1 w_1 + w_1 c_1 = 2$, and $w_1\mathbf{n}_0 = -\mathbf{n}_0 w_1$:

$$\Delta^2 = \beta^2(c_1\mathbf{n}_0)(c_1\mathbf{n}_0) = -\beta^2 c_1^2\mathbf{n}_0^2 = 0, \tag{20}$$

$$\Delta\kappa_1 = \frac{\beta}{2}(c_1\mathbf{n}_0)(c_1 w_1) = -\frac{\beta}{2}c_1^2\mathbf{n}_0 w_1 = 0, \tag{21}$$

$$\kappa_1\Delta = \frac{\beta}{2}(c_1 w_1)(c_1\mathbf{n}_0) = \frac{\beta}{2}c_1(2 - c_1 w_1)\mathbf{n}_0 = \beta c_1\mathbf{n}_0 - \frac{\beta}{2}c_1^2 w_1\mathbf{n}_0 = \beta c_1\mathbf{n}_0 = \Delta. \tag{22}$$

Consequently, expanding the square of the shifted observer κ_1' gives:

$$(\kappa_1')^2 = (\kappa_1 + \Delta)^2 = \kappa_1^2 + \kappa_1\Delta + \Delta\kappa_1 + \Delta^2 = \kappa_1 + \Delta + 0 + 0 = \kappa_1'. \tag{23}$$

Furthermore, the Pfaff time 1-form c_1 is preserved as a left eigen-one-form of κ_1':

$$c_1\kappa_1' = c_1(\kappa_1 + \beta c_1\mathbf{n}_0) = \frac{1}{2}c_1^2 w_1 + \beta c_1^2\mathbf{n}_0 = 0 \implies \bar{\kappa}_1' c_1 = (1 - \kappa_1')c_1 = c_1. \tag{24}$$

2. **Absence of Wigner Rotation (Commutativity of Non-Collinear Boosts):** To include spatial directionality without disturbing c_1 or w_1, note that the unit spacelike vector $\mathbf{u} := c_2 - c_3 \in \mathbb{G}(1,2)$ ($\mathbf{u}^2 = -1$) is orthogonal to both c_1 and w_1:

$$c_1 \cdot \mathbf{u} = c_1 \cdot c_2 - c_1 \cdot c_3 = \frac{1}{2} - \frac{1}{2} = 0, \qquad w_1 \cdot \mathbf{u} = 0 - 0 = 0.$$

Thus $\mathbf{u}$ anticommutes with c_1, w_1, and $\mathbf{n}_0$, which means $\mathbf{u}$ *commutes* with the idempotent $\kappa_1 = \frac{1}{2}c_1 w_1$! Defining the nilpotent boost rotor $T_{\mathbf{v}} := e^{\frac{1}{2}\mathbf{v}\mathbf{n}_0} = 1 + \frac{1}{2}\mathbf{v}\mathbf{n}_0$ for any spatial velocity $\mathbf{v} = v_x\mathbf{f}_1 + v_y\mathbf{f}_2 \in \mathbb{R}^{0,2}$, two successive boosts $T_{\mathbf{v}_1}$ and $T_{\mathbf{v}_2}$ compose via:

$$T_{\mathbf{v}_1}T_{\mathbf{v}_2} = \left(1 + \frac{1}{2}\mathbf{v}_1\mathbf{n}_0\right)\left(1 + \frac{1}{2}\mathbf{v}_2\mathbf{n}_0\right) = 1 + \frac{1}{2}(\mathbf{v}_1 + \mathbf{v}_2)\mathbf{n}_0 = T_{\mathbf{v}_1+\mathbf{v}_2}, \tag{25}$$

because $(\mathbf{v}_1\mathbf{n}_0)(\mathbf{v}_2\mathbf{n}_0) = -\mathbf{v}_1\mathbf{v}_2\mathbf{n}_0^2 = 0$. Unlike standard hyperbolic Lorentz rotors in $\mathbb{G}(1,3)$ where $e^{B_1/2}e^{B_2/2}$ produces a spatial Wigner rotation bivector $\mathbf{v}_1 \wedge \mathbf{v}_2$, the nilpotent radical $\mathbf{n}_0^2 = 0$ annihilates the cross-term identically.

3.4 Null Factorization and the Simplification of Singular Value Decomposition (SVD)

The algebraic verification of the shifted observer idempotent $\kappa_1' = \kappa_1 + \beta c_1\mathbf{n}_0$ in Section 3.3 reveals a striking computational advantage of the Universal Null Substrate over classical matrix analysis: **the elimination of artificial singular value distortions caused by nilpotent shears.**

The Classical Euclidean SVD Bottleneck: In standard linear algebra, when an operator A is non-symmetric, defective, or contains a nilpotent Jordan shear, its eigenvectors collapse and it cannot be orthogonally diagonalized by a single similarity transformation UAU^T. To recover a diagonal representation, classical Singular Value Decomposition (SVD) forces the operator into a Euclidean metric frame by forming the normal products A^TA and AA^T, yielding $A = U\Sigma V^T = \sum_k \sigma_k \mathbf{u}_k \otimes \mathbf{v}_k^T$ with orthonormal vectors $\mathbf{u}_k^2 = \mathbf{v}_k^2 = 1$.

Consider the 2×2 matrix representation of our shifted observer operator $\kappa_1' = \kappa_1 + \beta c_1\mathbf{n}_0$:

$$M(\kappa_1') = \begin{pmatrix} 1 & \beta \\ 0 & 0 \end{pmatrix}. \tag{26}$$

Although $M(\kappa_1')$ is an exact idempotent ($M^2 = M$) with eigenvalues $\lambda \in \{1, 0\}$, forming the normal matrix MM^T or M^TM destroys its idempotent structure:

$$M^TM = \begin{pmatrix} 1 & \beta \\ \beta & \beta^2 \end{pmatrix} \implies \sigma_1 = \sqrt{1+\beta^2}, \quad \sigma_2 = 0. \tag{27}$$

Because a Euclidean unit sphere cannot undergo a shear β without stretching its radius to $\sqrt{1+\beta^2}$, classical SVD is forced to introduce an irrational singular value $\sigma_1 = \sqrt{1+\beta^2}$ and rotate the right Euclidean basis by $\theta = \arctan(\beta)$:

$$\begin{pmatrix} 1 & \beta \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \sqrt{1+\beta^2} & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{1+\beta^2}} & \frac{\beta}{\sqrt{1+\beta^2}} \\ -\frac{\beta}{\sqrt{1+\beta^2}} & \frac{1}{\sqrt{1+\beta^2}} \end{pmatrix}. \tag{28}$$

This Euclidean procedure obscures the underlying geometry: it conflates a pure translational velocity shift β with a metric dilation $\sqrt{1+\beta^2}$ and destroys the nilpotency $\Delta^2 = 0$.

Direct Null Factorization in the QSNV Substrate: In the Quadratic Space of Null Vectors $V^{cw} = V_4^c \oplus V_4^w$, rank-1 dyads do not require outer products of row and column vectors; they are native geometric products $\frac{1}{2}c_k w_k$ of primitive null generators ($c_k^2 = w_k^2 = 0$, $c_k \cdot w_k = 1$). Rather than forming A^TA, we simply factor c_1 out on the left of κ_1':

$$\kappa_1' = \frac{1}{2}c_1 w_1 + \beta c_1\mathbf{n}_0 = \frac{1}{2}c_1\left(w_1 + 2\beta\mathbf{n}_0\right) =: \frac{1}{2}c_1 w_1'. \tag{29}$$

Theorem 3.3 (Invariance of the Null Dyad under Nilpotent Shears)**.** *Let $c_1 \in V_4^c$ and $w_1 \in V_4^w$ be dual null generators satisfying $c_1^2 = w_1^2 = 0$ and $c_1 \cdot w_1 = 1$, and let $\mathbf{n}_0$ be a degenerate null vector ($\mathbf{n}_0^2 = 0$) orthogonal to both c_1 and w_1 ($c_1 \cdot \mathbf{n}_0 = w_1 \cdot \mathbf{n}_0 = 0$). Then for any scalar $\beta \in \mathbb{R}$, the shifted vector*

$$w_1' := w_1 + 2\beta \mathbf{n}_0 \tag{30}$$

remains a pure null vector that preserves unit duality with c_1:

$$(w_1')^2 = 0 \qquad \text{and} \qquad c_1 \cdot w_1' = 1. \tag{31}$$

Consequently, the sheared operator $\kappa_1' = \frac{1}{2} c_1 w_1'$ is already in canonical rank-1 factored form with unit singular weight $\sigma_1 = 1$, requiring no square roots or normal products $A^T A$.

Proof. Expanding the geometric square of $w_1' = w_1 + 2\beta \mathbf{n}_0$ using $w_1^2 = 0$, $\mathbf{n}_0^2 = 0$, and $w_1 \mathbf{n}_0 + \mathbf{n}_0 w_1 = 2(w_1 \cdot \mathbf{n}_0) = 0$ gives:

$$(w_1')^2 = w_1^2 + 2\beta(w_1 \mathbf{n}_0 + \mathbf{n}_0 w_1) + 4\beta^2 \mathbf{n}_0^2 = 0 + 0 + 0 = 0.$$

Similarly, the inner product with c_1 evaluates immediately by linearity:

$$c_1 \cdot w_1' = c_1 \cdot (w_1 + 2\beta \mathbf{n}_0) = c_1 \cdot w_1 + 2\beta(c_1 \cdot \mathbf{n}_0) = 1 + 0 = 1.$$

Thus $\kappa_1' = \frac{1}{2} c_1 w_1'$ satisfies $(\kappa_1')^2 = \frac{1}{4} c_1 (2 - c_1 w_1') w_1' = \frac{1}{2} c_1 w_1' = \kappa_1'$ identically. □

This direct absorption of nilpotent shears into the primitive null generators $\{c_k, w_k\}$ complements and extends the matrix-free formulations of Singular Value Decomposition, polar decomposition, and generalized spectral decompositions in Clifford algebras recently developed by Shirokov [1], showing how the neutral phase space $\mathbb{G}(n, n)$ bypasses normal products $A^T A$ entirely along the lightcone.

4 Part IV: Physical Meaning: The Conformal Group $O(2, 4)$ and Twistors

4.1 The Conformal Group of Spacetime (Item 13)

In the 6-dimensional Conformal Spacetime Algebra $\mathbb{G}(2, 4) = \mathbb{G}(1, 3) \oplus^{\perp} \mathbb{G}(1, 1)$, let $\{e_1, f_1, f_2, f_3\}$ be the Hadamard basis of $\mathbb{G}(1, 3)$ from V_4^c, and let $\{\mathbf{n}_\infty, \mathbf{n}_o\}$ be the null pair spanning the adjoined hyperbolic plane $\mathbb{G}(1, 1)$ with $\mathbf{n}_\infty^2 = \mathbf{n}_o^2 = 0$ and $\mathbf{n}_\infty \cdot \mathbf{n}_o = 1$.

The $\binom{6}{2} = 15$ independent bivectors of $\mathbb{G}(2, 4)$ generate the rotor group $\mathrm{Spin}^+(2, 4)$, which double-covers the 15-parameter conformal group $SO^+(2, 4)$ of Minkowski spacetime via $X \mapsto RX\tilde{R}$:

1. **Lorentz Rotations and Boosts (6 generators):** Bivectors $B \in \mathbb{G}^2(1, 3)$, generating $R_L = e^{B/2}$.

2. **Spacetime Translations (4 generators):** Null bivectors $a \wedge \mathbf{n}_\infty$ for $a \in \mathbb{R}^{1,3}$, generating nilpotent translation versors:

$$T_a = e^{\frac{1}{2} a \mathbf{n}_\infty} = 1 + \frac{1}{2} a \mathbf{n}_\infty \qquad \text{since } (a\mathbf{n}_\infty)^2 = 0. \tag{32}$$

3. **Special Conformal Transformations (4 generators):** Null bivectors $b \wedge \mathbf{n}_o$ for $b \in \mathbb{R}^{1,3}$, generating $K_b = 1 + \frac{1}{2} b \mathbf{n}_o$.

4. **Dilations / Scale Transformations (1 generator):** The hyperbolic bivector $E = \mathbf{n}_\infty \wedge \mathbf{n}_o$ ($E^2 = +1$), generating $D_\lambda = e^{\frac{\lambda}{2} E} = \cosh(\frac{\lambda}{2}) + E \sinh(\frac{\lambda}{2})$.

4.2 Dirac's Six-Cone from the Null Substrate (Item 14)

In his famous 1936 paper *Wave Equations in Conformal Space*, Paul Dirac showed that Maxwell's equations and massless field equations in 4D Minkowski space become manifestly linear and invariant when lifted to the 5-dimensional projective null cone

$$\mathcal{N}^5 = \{X \in \mathbb{R}^{2,4} \setminus \{0\} \mid X^2 = 0\}/\mathbb{R}^\times, \tag{33}$$

where a point $x \in \mathbb{R}^{1,3}$ is represented by the null vector $X(x) = x + \frac{1}{2}x^2\mathbf{n}_\infty + \mathbf{n}_o$ ($X^2 = x^2 - x^2(\mathbf{n}_\infty \cdot \mathbf{n}_o) = 0$).

In standard presentations of Conformal Geometric Algebra (CGA), the two extra dimensions $\mathbb{R}^{1,1}$ are introduced as a purely mathematical trick to linearize spheres and translations. In the Universal Null Substrate, by contrast, $\mathbb{G}(2,4)$ is not an artificial embedding: because all spacetime vectors in V_4^c and V_4^w are already built from primitive null directions ($c_i^2 = w_i^2 = 0$), Dirac's six-cone is the native geometric habitat of the substrate bridge.

4.3 Penrose Twistors and Zero Residue Factorization (Item 15)

In Roger Penrose's twistor theory, points and null rays in complexified Minkowski space correspond to subspaces in a 4-dimensional complex space $\mathbb{T} \cong \mathbb{C}^4$ equipped with an Hermitian signature $(2,2)$ form, acted upon by $SU(2,2)$. Because $\mathrm{Spin}^+(2,4) \cong SU(2,2)$, spinors of our real Conformal Spacetime Algebra $\mathbb{G}(2,4)$ (and $\mathbb{G}(4,2)$) **are** Penrose twistors.

Disjoint Commuting Idempotents and Sylvester Decomposition: Rather than reverting to complex 4×4 matrices or orthonormal coordinates, we construct the minimal left ideals directly from the primitive null generators of V_4^c and V_4^w via Zero Residue Factorization (ZRF). While matched dyads $\kappa_k = \frac{1}{2}c_k w_k$ do not commute across different nodes ($[\kappa_1, \kappa_2] \neq 0$), elementary projectors defined on **disjoint index planes** $\{1,2\}$ and $\{3,4\}$ strictly commute. Specifically, define the cohesive projector pair on the $(1,2)$-null plane and the repulsive projector pair on the disjoint $(3,4)$-null plane, both satisfying the Sylvester minimal polynomial $x^2 - x = 0$:

$$I_1 := P_{12}^{\mathbf{c}} = c_1 c_2, \qquad \bar{I}_1 := \bar{P}_{12}^{\mathbf{c}} = 1 - c_1 c_2 = c_2 c_1, \tag{34}$$

$$I_2 := P_{34}^{\mathbf{w}} = -w_3 w_4, \qquad \bar{I}_2 := \bar{P}_{34}^{\mathbf{w}} = 1 + w_3 w_4 = -w_4 w_3. \tag{35}$$

Because $\{1,2\} \cap \{3,4\} = \emptyset$, the bridge rule $c_i \cdot w_j = \delta_{ij}$ reduces to strict anticommutativity $c_i w_j = -w_j c_i$ for $i \in \{1,2\}$ and $j \in \{3,4\}$. Consequently, I_1 and I_2 strictly commute ($[I_1, I_2] = [P_{12}^{\mathbf{c}}, P_{34}^{\mathbf{w}}] = 0$), and their Boolean products generate four mutually annihilating primitive idempotents $\Phi_1, \Phi_2, \Phi_3, \Phi_4$ partitioning the phase space ($\Phi_a \Phi_b = \delta_{ab}\Phi_a$, $\sum_{a=1}^4 \Phi_a = 1$):

$$\Phi_1 := P_{12}^{\mathbf{c}} P_{34}^{\mathbf{w}} = -(c_1 c_2)(w_3 w_4) = \left(\frac{1}{2} + c_1 \wedge c_2\right)\left(\frac{1}{2} - w_3 \wedge w_4\right), \tag{36}$$

$$\Phi_2 := P_{12}^{\mathbf{c}} \bar{P}_{34}^{\mathbf{w}} = (c_1 c_2)(1 + w_3 w_4) = \left(\frac{1}{2} + c_1 \wedge c_2\right)\left(\frac{1}{2} + w_3 \wedge w_4\right), \tag{37}$$

$$\Phi_3 := \bar{P}_{12}^{\mathbf{c}} P_{34}^{\mathbf{w}} = -(1 - c_1 c_2)(w_3 w_4) = \left(\frac{1}{2} - c_1 \wedge c_2\right)\left(\frac{1}{2} - w_3 \wedge w_4\right), \tag{38}$$

$$\Phi_4 := \bar{P}_{12}^{\mathbf{c}} \bar{P}_{34}^{\mathbf{w}} = (1 - c_1 c_2)(1 + w_3 w_4) = \left(\frac{1}{2} - c_1 \wedge c_2\right)\left(\frac{1}{2} + w_3 \wedge w_4\right), \tag{39}$$

where each branch carries the exact invariant scalar floor $\langle \Phi_a \rangle_0 = \frac{1}{4}$.

Twistor Geometry on Dirac's Six-Cone: The minimal left ideals $\mathcal{S}_a = \mathbb{G}(2,4)\Phi_a$ realize Penrose twistor space directly over the real numbers:

- **Substrate Annihilation:** Because $c_1^2 = 0$ and $w_4^2 = 0$, the primitive null generators act as coordinate-free nilpotent ladder operators on the ideal:

$$c_1\Phi_1 = 0 \qquad \text{and} \qquad \Phi_1 w_4 = 0. \tag{40}$$

- **Projective Incidence on the Six-Cone:** A spacetime point $X \in \mathbb{R}^{2,4}$ on Dirac's projective null cone ($X^2 = 0$) is incident with a twistor $\psi \in \mathcal{S}_1$ if and only if

$$X\psi = 0, \tag{41}$$

  replacing Penrose's complex coordinate relation $\omega^A = ix^{AA'}\pi_{A'}$ with a single geometric product.

- **Matrix-Free Evaluation:** Every twistor transformation, helicity projection, and conformal boost reduces to polynomial multiplication of c_i and w_j governed by $c_i c_j = 1 - c_j c_i$, $w_i w_j = -1 - w_j w_i$, and $c_i w_j = \delta_{ij} - w_j c_i$, completely bypassing 4×4 complex Dirac or $SU(2,2)$ matrices.

Concluding Perspective: From the single nilpotent axiom $n^2 = 0$ and linearity, the Universal Null Substrate generates the inner product, the symmetric Hadamard bases of $\mathbb{G}(1,3)$ and $\mathbb{G}(3,1)$, the metric-free observer idempotents $\kappa_k = \frac{1}{2}c_k w_k$, and the four mutually annihilating composite idempotents $\Phi_1, \ldots, \Phi_4$ of $\mathbb{G}(4,4)$—providing a unified, computationally elementary language for relativistic and quantum physics.

References

[1] D. S. Shirokov, "On Singular Value Decomposition and Polar Decomposition in Clifford Algebras," *Advances in Applied Clifford Algebras*, and related lecture series on matrix-free spectral decompositions in geometric algebras (HSE University, Moscow).

[2] D. Hestenes and G. Sobczyk, *Clifford Algebra to Geometric Calculus: A Unified Language for Mathematics and Physics*, D. Reidel, Dordrecht, 1984.

[3] G. Sobczyk, *Matrix Gateway to Geometric Algebra, Spacetime and Spinors*, Independent Publisher, 2019.

[4] G. Sobczyk, "Geometric Algebras of Light Cone Projective Graph Geometries," *Advances in Applied Clifford Algebras*, Vol. 34, Art. 7, 2024.

[5] G. Sobczyk and K. Tynski, "The Null Eightfold Way: Geometric Representation Theory and the Mass Gap within the $\mathbb{G}(4,4)$ Phase Space," Preprint, October 2026.