CLIFFORD ALGEBRA LECTURES

Garret Sobczyk[1]

Departamento de Fisico-Matematicas
Universidad de las Americas
Santa Catarina Mártir
72820 Cholula, Pue., México
sobczyk@udlapvms.pue.udlap.mx
FAX: (52-22) 29-2066

October 17, 1995

[1]I gratefully acknowledge the support given by INIP of the Universidad de las Americas, and CONACYT (The Mexican National Council for Science and Technology) grant 3803-E.

Chapter 1

Introduction to Geometric Algebras

Everybody knows that the real number field $\mathbb{R}$ can be extended to the complex number field $\mathbb{C} \equiv \mathbb{R}[i]$ by introducing an imaginary unit i with the property that $i^2 = -1$. But few mathematicians consider the extension of the real number system by a *unipotent* u defined by $u \neq \pm 1$ and $u^2 = 1$, perhaps because the resulting *hyperbolic number system* $\mathbb{U}$ is no longer a field but a commutative ring with unity.

The hyperbolic numbers are one of the simplest examples of Clifford's *geometric algebras*, and it is well worth the student's time to become familiar with its properties before going on to the study of more general geometric algebras. The intent of this lecture is to introduce the *unipodal number system*, and then the more general noncommutative geometric algebras of higher dimensions.

1.1 The Unipodal Number System

In the *standard basis* $\{1, u\}$ a *unipodal number* $w \in \mathbb{U}$ has the form $w = w_0 + uw_1$ where $u^2 = 1$ and $w_0, w_1 \in \mathbb{C}$. The *idempotent basis* $\{u_+, u_-\}$ is defined by

$$u_+ = \frac{1}{2}(1+u) \quad \text{and} \quad u_- = \frac{1}{2}(1-u),$$

and satisfies $u_+ + u_- = 1$ and $u_+ - u_- = u$. We say that u_+ and u_- are *idempotents* because $u_+^2 = u_+$ and $u_-^2 = u_-$, and they are *mutually annihilating* because $u_+u_- = 0$. Using the projective properties of the idempotent basis we can write

$$w = w(u_+ + u_-) = w_+u_+ + w_-u_-, \tag{1.1}$$

where $w_+ = w_0 + w_1$ and $w_- = w_0 - w_1$. Conversely, given $w = w_+u_+ + w_-u_-$ for $w_+, w_- \in \mathbb{C}$, we can recover the coordinates with respect to the standard basis by

$$w_0 = \frac{1}{2}(w_+ + w_-) \quad \text{and} \quad w_1 = \frac{1}{2}(w_+ - w_-). \tag{1.2}$$

The special properties of the idempotent basis make it particularly suitable for calculations. For example, the binomial theorem takes the very simple form

$$(w_+u_+ + w_-u_-)^k = (w_+)^k u_+^k + (w_-)^k u_-^k = (w_+)^k u_+ + (w_-)^k u_- \tag{1.3}$$

This formula is valid for *all* real numbers $k \in \mathbb{R}$, and not just the positive integers. For example, for $k = -1$ we find that

$$1/w = w^{-1} = (1/w_+)u_+ + (1/w_-)u_-,$$

a valid formula for the inverse of $w \in \mathbb{U}$, provided that $w_+w_- \neq 0$.

Indeed, the validity of (1.3) allows us to extend the definitions of *all* of the elementary functions to the elementary functions in the complex unipodal plane. If $f(w)$ is such a function for $w = w_+u_+ + w_-u_-$, we define

$$f(w) \equiv f(w_+)u_+ + f(w_-)u_- \tag{1.4}$$

provided that $f(w_+)$ and $f(w_-)$ are defined.

The hyperbolic numbers have been used to find the solution of the cubic equation and to derive simple properties of the Lorentzian geometry of the plane and special relativity [?].

There is an algebra isomorphism between complex 2×2 diagonal matrices and the unipodal numbers. We have the correspondence

$$w = w_+u_+ + w_-u_- \leftrightarrow \begin{pmatrix} w_+ & 0 \\ 0 & w_- \end{pmatrix}$$

As we shall see, the relationship between finite dimensional Clifford algebra and matrix algebra is that of an *algebra isomorphism:* Every finite dimensional Clifford algebra is isomorphic to a matrix algebra, and conversely, every matrix algebra is isomorphic to either a Clifford algebra or a subalgebra of a Clifford algebra, [**?**], [**?**].

1.2 Clifford Algebra Matrix Algebra Connection

We discovered at the end of the previous section that there is an algebra isomorphism between the unipodal numbers and diagonal 2×2 matrices. This observation leads to the question of whether it is possible to further extend the unipodal numbers in such a way that the extended number system is isomorphic to the full 2×2 matrix algebra. Of course, if we are to succeed we must introduce *noncommutative* elements.

Let us *split* or *factor* the unipotent element u into the product of two anticommuting units e and f satisfying the properties

$$u = ef = -fe \quad \text{and} \quad e^2 = 1 = -f^2; \tag{1.5}$$

we give $u = ef$ the interpretation of a *bivector* because it is the product of the two anticommuting *vectors* e and f. We say that that the geometric algebra $Cl_{1,1}$ is generated by the anticommuting vectors e and f, and as a linear space has the basis $\{1, e, f, ef\}$. A general element $w \in Cl_{1,1}$ can be written in the form $w = a_0 + a_1 u + e(b_0 + b_1 u)$, and also in the form

$$w = w(u_+ + u_-) = (a_0 + a_1)u_+ + (b_0 - b_1)eu_- + (b_0 + b_1)eu_+ + (a_0 - a_1)u_.$$

It may be shown that the correspondence

$$w \leftrightarrow \begin{pmatrix} a_0 + a_1 & b_0 - b_1 \\ b_0 + b_1 & a_0 - a_1 \end{pmatrix}$$

is the desired isomorphism.

These ideas have a straight forward generalization: Let us extend the real number system by assuming the existence of n commuting unipotent elements $u_1, u_2, \ldots, u_n$, obtaining the unipodal algebra $\mathbb{R}[u_1, u_2, \ldots, u_n]$. To exhibit the algebra isomorphism between this unipodal algebra and the $n \times n$ diagonal matrix algebra $M(2^n)$, we construct a set of 2^n mutually annihilating *primitive idempotents* u_{sgn} by taking the products of the 2^n sign combinations of the idempotents $u_{j\pm} \equiv \frac{1}{2}(1 \pm u_j)$. Thus, for example,

$$u_{++\ldots+} \equiv u_{1+}u_{2+}\ldots u_{n+},$$

and

$$u_{+-+\ldots+} \equiv u_{1+}u_{2-}u_{3+}\ldots u_{n+}.$$

A general element $w \in \mathbb{R}[u_1, \ldots, u_n]$ can now be expressed in the basis of primitive idempotents by

$$w = \sum_{sgn} w_{sgn} u_{sgn}$$

where the 2^n coefficients w_{sgn} are real scalars. The algebra isomorphism between $\mathbb{R}[u_1, \ldots, u_n]$ and diagonal matrices in $M(2^n)$ is given by

$$w \leftrightarrow \begin{pmatrix} w_{++\ldots+} & 0 & 0 & \ldots & 0 \\ 0 & w_{+-++\ldots+} & 0 & \ldots & 0 \\ \ldots & & & & \\ \ldots & 0 & \ldots & 0 & w_{--\ldots-} \end{pmatrix}$$

We can obtain the *neutral* Clifford algebra $Cl_{n,n}$ by splitting or factoring each of the unipotent elements u_j into a product of anticommuting unit vectors

$$u_j = e_j f_j = -f_j e_j, \quad \text{and} \quad e_j^2 = 1 = -f_j^2,$$

for $j = 1, 2, \ldots, n$. The following table gives the algebra isomorphism between the real Clifford algebra $Cl_{n,n}$ and the real matrix algebra $M(2^n)$:

$$\begin{pmatrix} u_{+\dots+} & e_1 u_{-+\dots+} & \dots & e_\lambda^{-1} u_{sgns} & \dots & e_{1\dots n}^{-1} u_{-\dots-} \\ e_1 u_{+\dots+} & u_{-+\dots+} & \dots & e_1 e_\lambda^{-1} u_{sgns} & \dots & e_1 e_{1\dots n}^{-1} u_{-\dots-} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ e_\lambda u_{+\dots+} & e_\lambda e_1 u_{-+\dots+} & \dots & u_{sgns} & \dots & e_\lambda e_{1\dots n}^{-1} u_{-\dots-} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ e_{1\dots k} u_{+\dots+} & e_{1\dots n} e_1 u_{-+\dots+} & \dots & e_{1\dots k} e_\lambda^{-1} u_{sgns} & \dots & u_{-\dots-} \end{pmatrix} \tag{1.6}$$

The Clifford numbers in the matrix make up a real 2^{2n}-dimensional basis of the Clifford algebra $Cl_{n,n}$; the position of each element corresponds to a matrix with a 1 in the same position and zeros everywhere else. The elements u_{sgn} are 2^n mutually annihiliating primitive idempotents, and the elements $e_1, \dots, e_n$ are orthonormal basis vectors with signature $+1$ generating $Cl_{n,0}$. The e_λ represents a product of the generating e_i's corresponding to the *ordered* index set λ, for example for $\lambda = \{1, 2, n\}$, $e_\lambda \equiv e_1 e_2 e_n$. The elements e_λ *conjugate commute* with the idempotent elements u_{sgn}, for example $e_{12n}\ u_{+++\dots++} = u_{--+\dots+-}\ e_{12n}$. A more detailed discussion of this isomorphism can be found in [**?**].

1.3 Geometric algebra

Let V be a not necessarily finite dimensional real vector space endowed with an inner product $v_1 \cdot v_2$ for vectors $v_1, v_2 \in V$. The associative *geometric algebra* $G(V)$ consists of all sums of products $v_1 v_2 \dots v_k$ of vectors in V subject to the following two principles of *geometric multiplication*:

- $v^2 = v \cdot v$ for all $v \in V$.
- If the vectors v_i are pairwise orthogonal, i.e., $v_i \cdot v_j = 0$ for $i \neq j$, then the geometric product $v_1 v_2 \cdots v_k$ is *skewsymmetric* over the interchange of any pair of vectors, and has the geometric interpretation of a *simple k-vector*.

More generally, a *k-vector* $A_k = B_k + C_k + \dots$ is the sum of simple k-vectors B_k, C_k, ... We are implicitly assuming that the addition and multiplication of elements in $G(V)$ satisfy all of the usual properties of the real numbers, except the commutative law of multiplication.[1]

The geometric algebra $I\!R_n \equiv G(I\!R^n)$ is constructed by taking all sums of geometric products of vectors in the n-dimensional euclidean vector space $I\!R^n$. The 2^n-dimensional geometric algebra $I\!R_n$ is a *graded* linear space in that

$$I\!R_n = < I\!R_n >_0 + < I\!R_n >_1 + \ldots + < I\!R_n >_n, \tag{1.7}$$

where $I\!R_n^k \equiv < I\!R_n >_k$ denotes the $\binom{n}{k}$-dimensional subspace of all k-vectors of $I\!R_n$. The geometric algebra $I\!R_n$ is an *extension* of the real number system $I\!R$ in-so-far as that $I\!R \equiv I\!R_n^0 \subset I\!R_n$. Note also that the vectors in $I\!R^n$ are identical to the subspace of 1-vectors, *i.e.*, $I\!R^n \equiv I\!R_n^1$.

To find the interpretation of the geometric product ab for arbitrary vectors $a, b \in I\!R^n$, we first write $b = b_\parallel + b_\perp$ where $b_\parallel$ and $b_\perp$ are the vector components of b parallel and perpendicular to a, respectively. Using the distributive property and employing the two principles of geometric multiplication, we calculate

$$ab = a(b_\parallel + b_\perp) = ab_\parallel + ab_\perp, \tag{1.8}$$

where $ab_\parallel \equiv < ab >_0$ is the *scalar* or 0-vector part and $ab_\perp \equiv < ab >_2$ is the *bivector* or 2-vector part of the product ab, respectively, *Figure 1.*

There is an even more basic form of this last identity:

$$ab = a \cdot b + a \wedge b, \tag{1.9}$$

where $a \cdot b \equiv \frac{1}{2}(ab + ba) = ab_\parallel$ is the commutative inner product and $a \wedge b \equiv \frac{1}{2}(ab - ba) = ab_\perp$ is the anticommutative *outer product* of the vectors a and b. Letting i denote an oriented unit bivector in the plane of a and b so that $i^2 = -1$, we can express the last identity in the form

$$ab = a \cdot b + a \wedge b = |a||b|(\cos\theta + i \sin\theta) = |a||b| \exp i\theta, \tag{1.10}$$

[1] If V is endowed with a more general *nondegenerate metric* instead of an inner product, we refer to $Cl(V)$ as the *Clifford algebra* on V with respect to this metric.

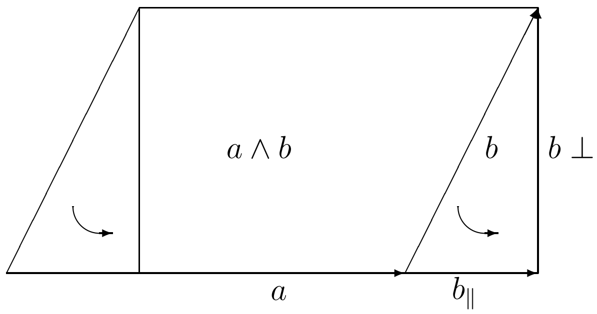

Figure 1.1: The outer product $a \wedge b = ab_{\perp}$ and is the *directed plane segment* represented either by the parallelogram with sides a and b, or by the rectangle with sides a and $b_{\perp}$.

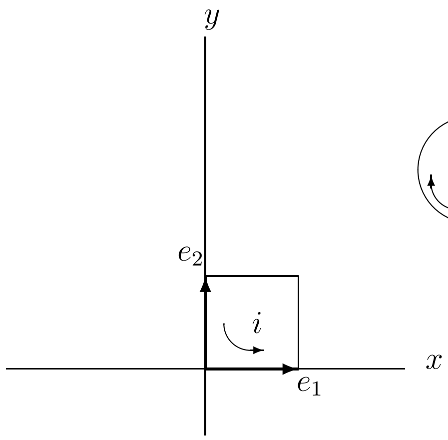

Figure 1.2: The unit bivector $i = e_1e_2 = -e_2e_1$ in the xy-plane is a *directed plane segment* with *orientation* indicated by the circular arrow. The disk with unit area represents the same bivector, but has *opposite* orientation.

called the *Euler formula* for the geometric multiplication of vectors, *Figure 2*. The Euler formula unites the inner and outer products of vectors into a single quantity, and shows that neither is more nor less fundamental than the other.

All other products are defined in terms of the basic geometric product. If $A_r \in I\!R_n^r$ and $B_s \in I\!R_n^s$ for $r, s \geq 1$, then the *inner product* of A_r and B_s is defined by

$$A_r \cdot B_s \equiv < A_r B_s >_{|r-s|}, \tag{1.11}$$

and the *outer product* by

$$A_r \wedge B_s \equiv < A_r B_s >_{r+s} . \tag{1.12}$$

The inner and outer products are linearly extended to arbitrary multivectors A and B in $I\!R_n$ with vanishing scalar parts.

Probably one of the biggest obstacle to the adoption of Clifford algebra methods in the general scientific community is the seemingly endless number of nontrivial algebraic identities that must be mastered. Many of these identities can be derived as simple consequences of the following basic identity for $a \in I\!R_n^1$ and $B_r \in I\!R_n^r$,

$$aB_r = a \cdot B_r + a \wedge B_r \tag{1.13}$$

for $r \geq 1$ and where

$$a \cdot B_r = \frac{1}{2}(aB_r + (-1)^{r+1} B_r a) = (-1)^{r+1} B_r \cdot a$$

and

$$a \wedge B_r = \frac{1}{2}(aB_r + (-1)^r B_r a) = (-1)^r B_r \wedge a.$$

We prove (1.13) in much the same way that we proved (1.9). Without loss of generality we can assume that B_r is a *simple* r-vector. Now write $a = a_\| + a_\perp$ where $a_\|$ is in the subspace of the simple r-vector B_r and $a_\perp$ is in the complementary subspace orthogonal to B_r. It then follows that

$$aB_r = (a_\| B_r + a_\perp B_r) = (-1)^{r+1} B_r a_\| + (-1)^r B_r a_\perp, \tag{1.14}$$

and

$$B_r a = B_r(a_\| + a_\perp) = B_r a_\| + B_r a_\perp. \tag{1.15}$$

Taking $(r-1)$ and $(r+1)$-vector parts of these identities gives

$$a \cdot B_r = a_\| B_r = (-1)^{r+1} B_r a_\|,$$

$$a \wedge B_r = a_\perp B_r = (-1)^r B_r a_\perp,$$

and also

$$B_r a = B_r \cdot a_\| + B_r \wedge a_\perp.$$

The proof is completed by taking the necessary sign combinations of these identities required to show (1.13).

As an application of these fundamental identities, we derive the important identity

$$a \cdot (b \wedge B_r) = (a \cdot b) B_r - b \wedge (a \cdot B_r). \tag{1.16}$$

We have

$$a \cdot (b \wedge B_r) = < a(b \wedge B_r) >_r = \frac{1}{2} < abB_r + (-1)^r aB_r b >_r$$

$$= (a \cdot b) B_r + \frac{1}{2} < -baB_r + (-1)^r aB_r b >_r$$

$$= (a \cdot b) B_r + \frac{1}{2} < -b(a \cdot B_r + a \wedge B_r) + (-1)^r (a \cdot B_r + a \wedge B_r) b >_r$$

$$= (a \cdot b) B_r - b \wedge (a \cdot B_r).$$

A complete discussion of these basic identities and others can be found in [?].

Chapter 2

Linear Transformations

Let V be a possibly infinite dimensional linear vector space over either the field of real or complex numbers. We shall first be interested in studying general properties of the endomorphism algebra $End_F(V)$ on V, where F is the field of real or complex numbers. The endomorphism algebra $End_F(V)$ exists independently of any further structure which may be assumed on V, such as a nondegenerate inner product induced by a quadratic form of arbitrary signature. Later we shall study the interelationship between the endomorphism algebra $End_F(V)$ and a metric structure $a \cdot b$ on V for vectors $a, b \in V$.

2.1 Structure of a Linear Operator

Let $End(\mathbb{C}^n)$ denote the endomorphism algebra of all complex linear operators on an n-dimensional complex vector space $\mathbb{C}^n$. We denote the identity operator in $End(\mathbb{C}^n)$ simply by 1. For a linear operator $a \in End(\mathbb{C}^n)$, let

$$\psi(\lambda) \equiv \prod_{i=1}^{r} (\lambda - \alpha_i)^{m_i}$$

be its *minimal polynomial*, having the distinct complex roots $\alpha_1, \ldots, \alpha_r \in \mathbb{C}$ with the respective *algebraic multiplicities* m_i. We order these roots so as to satisfy the inequalities $1 \le m_1 \le m_2 \le \ldots \le m_r$, where possibly the first $h \ge 0$ of them are equal to 1.

By the *structure equation* or *eigenprojector form* of the linear operator a with the minimal polynomial $\psi(\alpha)$, we mean

$$a = \sum_{i=1}^{r} (\alpha_i + q_i) p_i, \tag{2.1}$$

where the commutative operators p_i and q_j satisfy the properties

$$\{p_1 + \ldots + p_r = 1,\ p_i p_j = \delta_{ij} p_i,\ q_k^{m_k - 1} \neq 0 \text{ but } q_k^{m_k} = 0,\ p_k q_k = q_k\}. \tag{2.2}$$

The operators p_i make up a *partition of unity* and are *mutually annihilating idempotents.* The elements q_j are *nilpotents* with the respective *indexes* m_j. The nilpotents q_j are *projectively* related to the corresponding idempotents p_j and hence satisfy $p_j q_j = q_j$. We adopt the convention that $q_j \equiv 0$, for $j = 1, \ldots, h$. The commutative algebra generated by the p_i's and q_j's is just the *factor algebra* $\mathbb{C}\{m_1, m_2, \ldots, m_r\}$ (up to an isomorphism) of the principle ideal generated by the minimal polynomial $\psi(\lambda)$, [**?**].

The *Vandermonde determinant* is efficiently defined in terms of the column vector function $f(\lambda) = (1, \lambda, \lambda^2, \ldots, \lambda^{s-1})^T$ evaluated at the points $\lambda = \lambda_1, \lambda_2, \ldots, \lambda_s \in \mathbb{C}$,

$$V \equiv \det\{f(\lambda_1), f(\lambda_2), \ldots, f(\lambda_s)\} = \prod_{1 \le i < j \le s} (\lambda_j - \lambda_i). \tag{2.3}$$

The *normalized derivatives* $f^{(k)}(\lambda)$ of the column vector function $f(\lambda)$ are defined by

$$f_\lambda^{(k)} \equiv f^{(k)}(\lambda) = \frac{1}{k!} D_\lambda^k f(\lambda).$$

For example,

$$f_\lambda^{(1)} = (0, 1, 2\lambda, 3\lambda^2, \ldots, (s-1)\lambda^{(s-2)})^T,$$

and

$$f_\lambda^{(2)} = (0, 0, 1, 3\lambda, \ldots, \frac{(s-1)(s-2)}{2} \lambda^{(s-3)})^T.$$

With the above notation in hand, the determinant of the *generalized* Vandermonde matrix

$$W \equiv \{\underbrace{f_1 f_1^{(1)} \dots f_1^{(m_1-1)}}_{m_1 \ columns} \ \underbrace{f_2 f_2^{(1)} \dots f_1^{(m_2-1)}}_{m_2 \ columns} \ \dots \ \underbrace{f_r f_r^{(1)} \dots f_r^{(m_r-1)}}_{m_r \ columns}\}, \quad (2.4)$$

where $s = m_1 + m_2 + \dots + m_r$, becomes an elementary excercise of taking and evaluating the appropriate normalized partial derivatives of the equation (2.3), getting

$$\det W = \prod_{1 \le i < j \le r} (\lambda_j - \lambda_i)^{m_i m_j}. \quad (2.5)$$

We can now solve the structure equation (2.1) by a simple application of Cramer's rule to a system of operator equations [**?**]. Classically this structure theorem is known as the *generalized spectral decomposition* of a linear operator.

Theorem 1 *If $\psi(\alpha)$ is the minimal polynomial of the operator a, then a can be expressed in the eigenprojector form*

$$a = \sum_{j=1}^{r} (\alpha_j + q_j) p_j,$$

where the idempotent and nilpotent operators p_i and q_i, satisfying (2.2), are the unique polynomials in a of degree $< s = m_1 + \dots m_r$, specified by

$$p_i = \frac{\det\{\overbrace{f_1 \dots f_1^{(m_1-1)}}^{m_1} \dots \overbrace{f_a f_i^{(1)} f_i^{(2)} \dots f_i^{(m_i-1)}}^{m_i} \dots \overbrace{f_r \dots f_r^{(m_r-1)}}^{m_r}\}}{\det W},$$

and

$$q_i = \frac{\det\{\overbrace{f_1 \dots f_1^{(m_1-1)}}^{m_1} \dots \overbrace{f_i f_a f_i^{(2)} \dots f_i^{(m_i-1)}}^{m_i} \dots \overbrace{f_r \dots f_r^{(m_r-1)}}^{m_r}\}}{\det W},$$

where the $\det W$ *is given in (2.5).*

Proof. We begin by writing

$$a^0 \equiv 1 = \quad p_1 \quad + \quad p_2 \quad + \quad \dots \quad + \quad p_r,$$

and

$$a = (\alpha_1 + q_1)p_1 \; + \; (\alpha_2 + q_2)p_2 \; + \; \dots + \; (\alpha_r + q_r)p_r.$$

Using the assumed properties (2.2), we take successive powers of a to complete s powers of a:

$$a^2 = (\alpha_1 + q_1)^2 p_1 + (\alpha_2 + q_2)^2 p_2 + \dots + (\alpha_r + q_r)^2 p_r$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$a^{s-1} = (\alpha_1 + q_1)^{s-1} p_1 + (\alpha_2 + q_2)^{s-1} p_2 + \dots + (\alpha_r + q_r)^{s-1} p_r$$

The powers $(\alpha_j + q_j)^k$ can be easily computed, getting

$$(\alpha_j + q_j)^k = \alpha_j^k + \binom{k}{1} \alpha_j^{k-1} q_j + \dots + \binom{k}{m_j - 1} \alpha_j^{k-m_j+1} q_j^{m_j-1},$$

where $\binom{k}{j}$ are the usual binomial coefficients for $j \le k$, and $\binom{k}{j} \equiv 0$ for $j > k$. When each of these expansions is substituted back into the successive powers of a, we are led to a system of operator equations which are linear in

$$\{\overbrace{p_1, q_1, q_1^2, \dots q_1^{m_1-1}}^{m_1}, \dots, \overbrace{p_r, q_r, q_r^2, \dots, q_r^{m_r-1}}^{m_r}\}.$$

Just as for a system of linear equations, the system of *operator equations* is consistent and has a unique solution if the determinant $det(W)$ of the coefficient matrix W of the unknown operators is nonvanishing. But W is just the Vandermonde matrix (2.4) given by

$$\begin{pmatrix} 1 & 0 & \dots & 1 & \dots & 1 & 0 & \dots \\ \alpha_1 & 1 & \dots & \alpha_j & \dots & \alpha_r & 1 & \dots \\ \vdots & & & & \vdots & & & \\ \alpha_1^{s-1} & (s-1)\alpha_1^{s-2} & \dots & \alpha_j^{s-1} & \dots & \alpha_r^{s-1} & (s-1)\alpha_r^{s-2} & \dots \end{pmatrix}$$

The determinant (2.5) of this matrix,

$$det(W) = \prod_{i<j}(\alpha_j - \alpha_i)^{m_i m_j} \neq 0,$$

since the roots α_i for $i = 1, \ldots, r$ are distinct. Applying Cramer's rule to this system of linear operator equations, gives the operator unknowns p_i, q_j in terms of polynomials in a as specified in the statement of the theorem.

Q.E.D.

The novel proof of this basic structure theorem given above is apparently missing in the literature. It has many consequences at a basic level, making it necessary to re-examine the proofs of many theorems in linear and multilinear algebra [**?**],[**?**],and [**?**].

2.2 Isometries

The power of geometric algebra as a tool in linear algebra is amply illustrated by the study of skewsymmtric transformations and their corresponding isometries. We shall lay down the basic ideas of this theory, following closely the seminal work of Marcel Reisz [**?**].

Let V be an n-dimensional vector space provided with a nondegenerate metric $x \cdot y$. We assume we have constructed a pair of *reciprocal bases* $\{e_i\}$ and $\{e^j\}$ satisfying the defining properties that $e_i \cdot e^j = \delta_i^j$. Let $f \in End(V)$ be a linear endomorphism on V.

Definition 1 *The endomorphism f is said to be an isometry of the metric $x \cdot y$ if $f(x) \cdot f(y) = x \cdot y$ for all $x, y \in V$.*

Closely connected to the idea of an isometry is the idea of a *skewsymmetric* transformation:

Definition 2 *An operator $h(x)$ is skewsymmetric on V relative to the metric $x \cdot y$ if*

$$h(x) \cdot y = x \cdot h^{\dagger}(y) = -x \cdot h(y)$$

for all $x, y \in V$.

It immediately follows from Definition 2 that $h(x)$ is skewsymmetric if and only if its *adjoint transformation* $h^\dagger(x) = -h(x)$ for all $x \in V$. A skewsymmetric transformation h has the important property that $h^k(x) \cdot y = x \cdot (-h)^k(y)$, from which it follows by a simply argument that $\exp(h)x \cdot y = x \cdot \exp(-h)y$. If we now replace y by $\exp(h)y$, we easily find that

$$\exp(h)x \cdot \exp(h)y = x \cdot y,$$

which establishes that for any skewsymmetric h, $\exp(h)$ is an isometry. If t is a continuous real (or complex) parameter, then we say that $g_t = \exp(th)$ is a one-parameter Abelian group of isometries continuously connected to the identity 1 ($t = 0$). Thus, to every skewsymmetric transformation there belongs a one-parameter group of isometries, and conversely.

Bivectors in Clifford algebra arise in the study of isometries of a given metric because of the one-to-one relationship between bivectors and skewsymmetric transformations: Each skewsymmetric transformation $h(x)$ can be written in the form $h(x) = x \cdot F$, where the bivector F is defined in terms of the vector derivative by $F \equiv \frac{1}{2}\partial_v \wedge h(v)$.[1]

We now give the theorem that specifies the exact relationship between isometries, skewsymmetric transformations and bivectors [**?**].

Theorem 2 *Let $g_t x = \exp(th)x$ be the one parameter group generated by the skewsymmetric transformation $h(x) = x \cdot F$, then $g_t x = \exp(th)x = \exp(-\frac{1}{2}tF)x\exp(\frac{1}{2}tF)$.*

Proof: Take and evaluate successive derivatives with respect to t at $t = 0$ of both sides of

$$\exp(th)x = \exp(-\frac{1}{2}tF)x\exp(\frac{1}{2}tF),$$

and check the equality of the resulting expressions.

Q.E.D.

[1]The vector derivative ∂_x is defined by

$$\partial_x \equiv \sum_{i=1}^{n} e^i \frac{\partial}{\partial x^i},$$

where for $x = \sum x^i e_i$, $\frac{\partial x}{\partial x^i} = e_i$.

2.3 Minimal Polynomials

The minimal polynomial of a skewsymmetric operator is an important tool for studying its properties. We consider here the minimal polynomials of skewsymmetric operators in the Clifford algebras $Cl_{1,3}$ and $Cl_{2,2}$, although the ideas can be generalized to skewsymmetric operators in Clifford algebras of arbitrary signatures.

Consider first the skewsymmetric operator $f(x) = x\cdot(e_0\wedge e_1+e_2\wedge e_3)$. With respect to the orthonormal basis e_0, e_1, e_2, e_3, we calculate $f(e_0) = e_1$, $f(e_1) = e_0$, $f(e_2) = -e_3$, and $f(e_3) = e_2$, from which it follows that the matrix of f with respect to this basis is

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

The minimal polynomial of the subspace determined by the bivector $e_0\wedge e_1$ is clearly $(\lambda-1)(\lambda+1)$, and that of the subspace of the bivector $e_2 \wedge e_3$ is $\lambda^2 + 1$. It follows that the minimal polynomial of f is $(\lambda^2 - 1)(\lambda^2+1)$. The bivector $F = e_0\wedge e_1+e_2\wedge e_3$ is the sum of two commuting orthogonal blades $e_0 \wedge e_1$ and $e_1 \wedge e_2$; the blade $e_0 \wedge e_1$ determines the plane of an *hyperbolic rotation*, and the blade $e_2 \wedge e_3$ determines the plane of an *eliptic rotation*. Conversely, if the minimal polynomial of a skewsymmetric transformation in $Cl_{1,3}$ is of the form

$$\psi(\lambda) = (\lambda^2 + \alpha^2)(\lambda^2 - \beta^2)$$

where $\alpha > 0$ and $\beta > 0$, it follows as a consequence of Theorem 1 that F is the sum of two commuting orthogonal blades which determine hyperbolic and eliptic rotations.

Consider now the skewsymmetric operator $f(x) = x \cdot (\overline{n} \wedge n)$ determined by the *parabolic* null bivector $\overline{n} \wedge n$, where $n \equiv e_0 + e_3$ and $\overline{n} \equiv \frac{1}{2}(e_0 - e_3)$ are null vectors. Calculating $f(e_1) = 0 = f(n)$, $f(e_2) = n$, and $f(\overline{n}) = e_2$, we find that the matrix of f with respect to the basis $e_1, n, e_2, \overline{n}$ is

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

so the minimal polynomial of f is λ^3. It follows that a parabolic bivector takes up 3 dimensions, so the corresponding skewsymmetric transformation can have but one invariant plane.

The difference in the Euclidean and Lorentz cases is due to the possibility of a factor λ^3 in the minimal polynomial. A factor of λ^3 in the minimal polynomial reveals the presence of a *parabolic rotation* in the plane of a *null bivector*, but the decomposition of an arbitrary bivector F into a sum of commuting orthogonal simple bivectors still remains valid for the reasons given above.

The decomposition of an arbitrary bivector F into a sum of commuting orthogonal simple bivectors fails in the case of the *ultra hyperbolic* metric with the signature $+ \; + \; - \; -$, as the following example shows. Take

$$n_1 = \frac{e_1 + f_1}{\sqrt{2}}, \; n_2 = \frac{e_2 + f_2}{\sqrt{2}}, \; n_3 = \frac{e_2 - f_2}{\sqrt{2}}, \; n_4 = \frac{e_1 - f_1}{\sqrt{2}}.$$

Then for $x = \sum x^j n_j$ and $y = \sum y^j n_j$, we have $x \cdot y = x^1 y^4 + x^2 y^3 + x^3 y^2 + x^4 y_1$.

Define the skewsymmetric transformation $h(x) = x \cdot F$ for

$$F \equiv n_4 \wedge (n_1 + n_2) + n_3 \wedge n_2 = e_1 f_1 + e_2 f_2 + n_4 n_2,$$

so that $h(n_1) = n_1 + n_2$, $h(n_2) = n_2$, $h(l_3) = -n_3 - n_4$, and $h(n_4) = -n_4$. The vectors n_2 and n_4 are eigenvectors with the respective minimal polynomials $(\lambda - 1)$ and $(\lambda + 1)$. The respective minimal polynomials of n_1 and n_3 are $(\lambda - 1)^2$ and $(\lambda + 1)^2$. It follows that the minimal polynomial of h is $\psi(\lambda) = (\lambda - 1)^2 (\lambda + 1)^2$. The bivectors $n_1 n_2$, $n_3 n_4$, and $n_2 n_4$ are eigenbivectors with the corresponding minimal polynomials $(\lambda - 1)^2$, $(\lambda + 1)^2$, and $(\lambda^2 - 1)$. Any other bivectors have minimal polynomials of degree ≥ 3 and hence are not invariants of h. But no two of these bivectors are orthonormal, so a orthogonal decomposition of h is impossible, [**?**, p. 171].

2.4 Lie Algebras of Bivectors

We will study here skewsymmetric mappings of the neutral Clifford algebra $Cl_{n,n}$. Following the notation in [**?**], we choose as our basis $e_i, \overline{e}_j$, where $e_i \cdot e_j = \delta_{ij} = -\overline{e}_i \cdot \overline{e}_j$, and $e_i \cdot \overline{e}_j = 0$ for $i, j = 1, 2, \ldots n$. We will be particularly interested in the subalgebras $N = gen\{n_1, n_2, \ldots, n_n\}$ and $\overline{N} = gen\{\overline{n}_1, \ldots, \overline{n}_n\}$ generated by by the null vectors $n_i = e_i + \overline{e_i}$ and $\overline{n}_i = e_i - \overline{e}_i$.

It is well known that the bivectors in $Cl^2_{n,n}$ form the Lie algebra $so_{n,n}$ of the Lie group $SO_{n,n}$ with the Lie (bracket) product

$$b_1 \otimes b_2 \equiv \frac{1}{2}[b_1, b_2] = \frac{1}{2}(b_1 b_2 - b_2 b_1)$$

for $b_1, b_2 \in Cl^2_{n,n}$. Consider now the Lie subalgebra gl_n of $Cl^2_{n,n}$ generated by the n^2 bivectors

$$gl_n = \; gen\{b_{ij} = e_j \wedge n_i + \frac{1}{2} n_i \wedge n_j\} \tag{2.6}$$

for $i, j = 1, 2, \ldots n$. The structure constants of this Lie algebra are easily checked to be

$$b_{ij} \otimes b_{kl} = \delta_{il} b_{kj} - \delta_{jk} b_{il}.$$

We shall show that gl_n is the Lie algebra of bivectors for the general linear group GL_n on N.

Let us write a general element $B \in gl_n$ in the form $B = \sum_{ij} a_{ij} b_{ij}$. The skewsymmetric mapping $T : N \longrightarrow N$ of this bivector, expressed in terms of the basis $\{n\} \equiv \{n_1, n_2, \ldots n_n\}$, is

$$T(x) = x \cdot B = \{n\} x_{\{n\}} \cdot B = \{n\} T_{\{n\}} x_{\{n\}} = \{n \cdot B\} x_{\{n\}} \tag{2.7}$$

where the vector $x = \{n\} x_{\{n\}} \in N$ has the *column* scalar components $x_{\{n\}} = \begin{pmatrix} x_1 & x_2 & \ldots & x_n \end{pmatrix}^\dagger$, and $T_{\{n\}} \equiv (\beta_{ij})$ is the $n \times n$ *matrix* of the transformation $x \cdot B$ with respect to the basis $\{n\}$.

For $x \in N$ and $A, B \in gl_n$ the *Jacobi identity* takes the form

$$(x \otimes A) \otimes B - (x \otimes B) \otimes A = x \otimes (A \otimes B).$$

If $A_{\{n\}}$ and $B_{\{n\}}$ are the matrices of the skew symmetric mappings $x \cdot A$ and $x \cdot B$, *i.e.*,

$$\{n\}A_{\{n\}} = \{n \cdot A\} \quad \text{and} \quad \{n\}B_{\{n\}} = \{n \cdot B\},$$

then we calculate

$$\{n\}A_{\{n\}}B_{\{n\}} = \{n \cdot A\}B_{\{n\}} = \{(n \cdot A) \cdot B\}.$$

Applying the Jacobi identity, we can then easily show that

$$\{n\}[A_{\{n\}}, B_{\{n\}}] = \{(n \cdot A) \cdot B - (n \cdot B) \cdot A\} = \{n \cdot (A \otimes B)\}. \quad (2.8)$$

But the $n \times n$ matrices under the Lie bracket product make up precisely the classical Lie algebra of the general linear group, [**?**].

One final observation: We can *project* the Lie algebra of bivectors gl_n onto the Lie subalgebra $so_{p,q}$, for $p+q = n$, of the *special orthogonal group* $SO_{p,q}$ by using the projection operator $P_{p,q} : N \longrightarrow Cl_{p,q}$ defined by

$$P_{p,q}X = E^{-1}(E \cdot X) \quad (2.9)$$

where $E \equiv e_1 \wedge \cdots \wedge e_p \wedge \overline{e}_{p+1} \wedge \ldots \wedge \overline{e}_n$,
for $X \in Cl_{p,q}$.

Chapter 3

Directed Integration

- Simplices and Chains
- Directed Integration
- Boundary Theorem
- Residue Theorem
- Riemannian Geometry

The purpose of this chapter is to introduce the reader to the basic ideas of *simplicial calculus*. For reasons of clarity, the proofs offered here are for the lower dimensions, their generalization to higher dimensions being self-evident. A discussion of some of the basic ideas of Riemannian geometry is included. The reader may wish to refer to [**?**] and [**?**] for more details and more general proofs.

3.1 Simplices and Chains

Let $\{a_0, a_1, \ldots, a_k\}$ be an ordered set of points in $I\!R^n$.

An *oriented k-simplex* is determined by an ordered set of $k+1$ vertices

$$\{a\}_{(k)} \equiv \{a_0, a_1, \ldots, a_k\} = \{a |\ a = \sum_{\mu=0}^{k} t_\mu a_\mu\}, \tag{3.1}$$

where $\sum_{\mu=0}^{n} t_\mu = 1$, and $0 \le t_\mu \le 1$, are the *barycentric coordinates* of a. Interchanging two vertices of $\{a\}_{(k)}$ changes its orientation:

$$\{a_0, \dots, a_i, \dots, a_j, \dots a_k\} = -\{a_0, \dots, a_j, \dots, a_i, \dots a_k\}. \tag{3.2}$$

The *Boundary* of $\{a\}_{(k)}$ is the *chain* of $(k-1)$-simplices

$$\partial\{a\}_{(k)} = \sum_{i=1}^{k} (-1)^i \{\check{a}_i\}_{(k-1)} \tag{3.3}$$

where $\{\check{a}_i\}_{(k-1)}$ is the $(k-1)$-simplex

$$\{\check{a}_i\}_{(k-1)} = \{a_0, a_1, \dots \check{a}_i \dots, a_k\},$$

formed by deleting the marked vertice $\check{a}_i$. For the k-simplices, for $k = 0, 1, 2, 3$, we have

$$\partial\{a\}_{(0)} = \{\},$$

$$\partial\{a\}_{(1)} = \{a_1\} - \{a_0\},$$

$$\partial\{a\}_{(2)} = \{a_1, a_2\} - \{a_0, a_2\} + \{a_0, a_1\},$$

and

$$\partial\{a\}_{(3)} = \{a_1, a_2, a_3\} - \{a_0, a_2, a_3\} + \{a_0, a_1, a_3\} - \{a_0, a_1, a_2\}.$$

A basic result of *homology theory* is

$$\partial^2\{a\}_{(k)} = \{\}, \tag{3.4}$$

i.e, the boundary of the boundary of a k-simplex is the empty set. For example

$$\partial^2\{a_0\} = \partial\{\} = \{\},$$

$$\partial^2\{a_0, a_1\} = \partial\{a_1\} - \partial\{a_0\} = \{\} - \{\} = \{\},$$

$$\partial^2\{a_0, a_1, a_3\} = \partial\{a_1, a_2\} - \partial\{a_0, a_2\} + \partial\{a_0, a_1\}$$

$$= \{a_2\} - \{a_1\} - \{a_2\} + \{a_0\} + \{a_1\} - \{a_0\} = \{\}.$$

The *directed content* of the k-simplex $\{a\}_{(k)}$, is the k-vector

$$a_{(k)} \equiv \frac{1}{k!}(a_1 - a_0) \wedge (a_2 - a_1) \wedge \ldots \wedge (a_k - a_{k-1}), \tag{3.5}$$

for $k \geq 1$. We also write $Dir\{a\}_{(k)} = a_{(k)}$. For $k = 0$ and $k = 1$, we define

$$Dir\{a_0\} \equiv 1, \quad Dir\{\} \equiv 0.$$

The directed content of $\{a\}_{(k)}$ can also be expressed by

$$a_{(k)} = \frac{1}{k!}(a_1 - a_0) \wedge (a_2 - a_0) \wedge \ldots \wedge (a_k - a_0). \tag{3.6}$$

For example,

$$a_{(3)} = \frac{1}{3!}(a_1 - a_0) \wedge (a_2 - a_1) \wedge (a_3 - a_2)$$

$$= \frac{1}{3!}(a_1 - a_0) \wedge (a_2 - a_0 + a_0 - a_1) \wedge (a_3 - a_2)$$

$$= \frac{1}{3!}(a_1 - a_0) \wedge (a_2 - a_0) \wedge (a_3 - a_0 + a_0 - a_2)$$

$$= \frac{1}{3!}(a_1 - a_0) \wedge (a_2 - a_0) \wedge (a_3 - a_0).$$

The *complimentary k-simplex* $\{a\}^{(k)}$ of the simplex $\{a\}_{(k)}$ at the point a_0 is defined by

$$\{a\}^{(k)} = \{a_0, a^1, \ldots, a^k\} \tag{3.7}$$

where the vertices $a^1, a^2, \ldots, a^k$ of the k-simplex $\{a\}^{(k)}$ are chosen to satisfy the conditions that $(a^i - a_0) \wedge a_{(k)} = 0$ and $(a_i - a_0) \cdot (a^j - a_0) = \delta_{ij}$ for $i, j = 1, 2, \ldots, k$. In other words, the sets of edge vectors $\{a_i - a_0\}$ and $\{a^j - a_0\}$ make up a pair of *reciprocal frames* with respect to the k-vector $a_{(k)}$. It is natural to define

$$a^{(k)} \equiv (k!)^2 Dir(\{a\}^{(k)}) = k!(a^k - a_0) \wedge \ldots \wedge (a^1 - a_0)$$

so that $a^{(k)} a_{(k)} = 1$. Thus, the k-vector $a^{(k)}$ equals $(k!)^2$ times the directed content of the complimentary simplex $\{a\}^{(k)}$.

Theorem 3 *The directed content of the boundary of a k-simplex is zero, i.e.,*

$$Dir\{\partial\{a\}_{(k)}\} = 0. \tag{3.8}$$

Proof: We give the proof here for $k = 0, 1, 2, 3$.
For $k = 0$, we have $Dir(\partial\{a\}) = Dir\{\} = 0$.
For $k = 1$, we have

$$Dir[\partial\{a_0, a_1\}] = Dir\{a_1\} - Dir\{a_0\} = 0.$$

For $k = 2$, we have

$$Dir[\partial\{a_0, a_1, a_2\}] = Dir\{a_1, a_2\} - Dir\{a_0, a_2\} + Dir\{a_0, a_1\}$$

$$= a_2 - a_1 - a_2 + a_0 + a_1 - a_0 = 0.$$

For $k = 3$, we have

$$\partial\{a\}_{(3)} = \{a_1, a_2, a_3\} - \{a_0, a_2, a_3\} + \{a_0, a_1, a_3\} - \{a_0, a_1, a_2\},$$

so that

$$Dir[\partial\{a\}_{(3)}] = \frac{1}{2}\{(a_2 - a_1) \wedge (a_3 - a_1) - (a_2 - a_0) \wedge (a_3 - a_0)$$

$$+(a_1 - a_0) \wedge (a_3 - a_0) - (a_1 - a_0) \wedge (a_2 - a_0)\}$$

$$= \frac{1}{2}\{(a_3-a_0-a_1+a_0)\wedge(a_2-a_0)+(a_2-a_1)\wedge(a_3-a_1)+(a_1-a_0)\wedge(a_3-a_0)\}$$

$$= \frac{1}{2}\{(a_3-a_1)\wedge(a_2-a_0-a_2+a_1)+(a_1-a_0)\wedge(a_3-a_0)\}$$

$$= \frac{1}{2}\{(a_3-a_1)\wedge(a_1-a_0)+(a_1-a_0)\wedge(a_3-a_0)\} = 0.$$

Q.E.D.

The *directed integral* of a geometric algebra valued function $F(x)$ on an oriented k-surface S is defined by

$$\int_S d^k x F = \lim_{N\to\infty} \sum_{j=1}^{N} \Delta^k x_j F(x_j), \tag{3.9}$$

where $\Delta^k x_j$ is the directed k-vector contents of the k-simplex $\{\Delta^k x_j\}$ of S at x_j, and as $N \to \infty$ each of the k-simplices $\{\Delta^k x_j\}$ shrink to a single point on S in an appropiate manner.[1] Each partition divides up the surface S into a set of approximating k-simplices,

$$S \simeq \sum_{j=1}^{N} \{\Delta^k x_j\}$$

Example 1 *Let* $S = \sum_{i=1}^{N}\{a_i\}_{(0)}$. *Then, since* $Dir(\{a_i\}) = 1$ *for each* i*, it follows that*

$$\int_S d^0 x = \sum_i \; Dir(\{a_i\}) \; = N.$$

The following example shows that the the concept of a directed integral is compatible with the concept of the directed content of a simplex.

[1]More specifically, we require that the ratios of the radii of the circumscribing and inscribing k-spheres containing each of the k-simplices $\{\Delta^k x_j\}$ are bounded as $N \to \infty$.

Example 2 $\int_{\{a\}_{(k)}} d^k x = \ Dir(\{a\}_{(k)}) = a_{(k)} =$

$$= \frac{1}{k!}(a_1 - a_0) \wedge (a_2 - a_1) \wedge \ldots \wedge (a_k - a_{k-1}).$$

Example 3 $\int_{\{a\}_{(1)}} dx \ x \cdot b = \frac{1}{2}(a_1 - a_0)(a_0 + a_1) \cdot b.$

Proof. Let the line $x(t)$ joining the points a_0 and a_1 be parameterized by

$$x(t) = a_0 + t(a_1 - a_0), \quad \text{so that} \quad dx = dt(a_1 - a_0),$$

where $0 \leq t \leq 1$. We find

$$\int_{\{a\}_{(1)}} dx \ x \cdot b = \int_0^1 dt(a_1 - a_0)[a_0 + t(a_1 - a_0)] \cdot b$$

$$= (a_1 - a_0) \int_0^1 [a_0 \cdot b + t(a_1 - a_0) \cdot b] dt$$

$$= (a_1 - a_0) a_0 \cdot b + 1/2(a_1 - a_0)(a_1 - a_0) \cdot b$$

$$= \frac{1}{2}(a_1 - a_0)(a_0 + a_1) \cdot b.$$

Q.E.D.

Example 4 $\int_{\partial\{a\}_{(1)}} d^0 x \ x \cdot b = (a_1 - a_0) \cdot b.$

Proof. Recalling that $Dir\{a\} = 1$, we find that

$$\int_{\partial\{a_0, a_1\}} d^0 x \quad x \cdot b = \int_{\{a_1\} - \{a_0\}} d^0 x \ x \cdot b$$

$$= \int_{\{a_1\}} d^0 x \ x \cdot b - \int_{\{a_0\}} d^0 x \ x \cdot b = (a_1 - a_0) \cdot b.$$

Q.E.D.

Generalizing example 3 leads to

$$\int_{\{a\}_{(k)}} d^k x \ x \cdot b = \frac{1}{k+1} a_{(k)}(a_0 + \ldots + a_k). \tag{3.10}$$

Similarly, the generalization of example 4 is

$$\int_{\partial\{a\}_{(k)}} d^{k-1} x \ x \cdot b = (-1)^{k+1} a_{(k)} \cdot b. \tag{3.11}$$

Integral Definition of $\partial_x F(x)$ on S

Let S be a k-surface, x_0 an interior point of S, and $\{x\}_{(k)}$ a k-simplex in S at x_0. Even through $\{x\}_{(k)}$ may be small and a good approximation to S at the point x_0, we are only guaranteed that the vertices of $\{x\}_{(k)}$ will be in S (since S may have curvature at x_0). This poses problems in regard to multivector-valued functions $F(x)$ defined on S; we would like such functions to be defined locally on simplices which approximate S at the point $x_0 \in S$. This leads us to define the *affine approximation* to $F(x)$ on the simplex $\{x\}_{(k)}$ in S at x_0.

Definition 3 *By the affine approximation to $F(x)$ on the simplex $\{x\}_{(k)}$ in S at x_0, we mean the function*

$$f(x) = F(x_0) + \sum_{i=1}^{k} (x - x_0) \cdot (x^i - x_0)(F(x_i) - F(x_0)),$$

where $x^i - x_0$ are edge vectors of the complimentary simplex $\{x\}^{(k)}$ defined earlier, and $x \in \{x\}_{(k)}$.

From definition 3, it is seen that $f(x)$ agrees with $F(x)$ on the vertices of $\{x\}_{(k)}$, and is the affine approximation to $F(x)$ for points x in the interior of $\{x\}_{(k)}$.

Evidently, there is a close connection between the affine approximation to $F(x)$ at x_0 and *Taylor's theorem* at x_0. We give Taylor's theorem stated as a Lemma below for a function $F(x)$ differentiable in an open set U containing the point x_0 in the *flat* euclidean space $\mathbb{R}^k$.

Lemma 1 *If $F(x)$ is a continuous and differentiable geometric-valued function at the point $x = x_0 \in \mathbb{R}^k$, then*

$$F(x) = F(x_0) + (x - x_0) \cdot \nabla_{x_0} F(x_0) + |x - x_0| E(x, x - x_0),$$

where $E(x, y)$ is continuous at $x = x_0$ and linear in y, $(x - x_0) \cdot \nabla_{x_0}$ is the ordinary directional derivative in the direction of the vector $x - x_0$, and ∇_{x_0} is the gradient of $\mathbb{R}^k$.

The advantage of definition 1 over lemma 1 is that it is valid for any k-surface embedded in a higher dimensional linear space, whereas lemma 1 makes sense only in the euclidean space $I\!R^k$. Using definition 1, we now define the *vector derivative* $\partial_x F(x)$ at the point $x = x_0 \in S$.

Definition 4 *For* $k \geq 1$*,*

$$\partial_{x_0} F(x_0) = (-1)^{k+1} \lim_{\Delta^k x_0 \to 0} \frac{1}{\Delta^k x_0} \int_{\partial\{\Delta^k x_0\}} d^{k-1}x \ f(x),$$

where $f(x)$ *is the affine approximation to* $F(x)$ *at* $x = x_0$*.*

Definition 4 is the natural generalization of the ordinary one dimensional derivative at a point on the x-axis, to the vector derivative at a point on a k-surface. The following theorem shows that the nabla operator ∇_x is equivalent to the vector derivative ∂_x in $I\!R^k$.

Theorem 4 *If* $F(x)$ *is differentiable at* $x = x_0 \in I\!R^k$*, then* $\nabla_x F(x) = \partial_x F(x)$*.*

Proof: We will only outline the proof, freely using the fact that in $f(x) \simeq F(x)$ in small neighborhoods of the point x_0. By Taylor's Theorem in $I\!R^k$,

$$F(x) = F(x_0) + (x - x_0) \cdot \nabla_{x_0} F(x_0) + |x - x_0| E(x, x - x_0). \quad (3.12)$$

Using this, with the help of (3.11), we calculate

$$(-1)^{k+1} \partial_{x_0} F(x_0) = \lim_{\Delta^k x_0 \to 0} \frac{1}{\Delta^k x_0} \int_{\partial \Delta^k x_0} d^{k-1}x f(x)$$

$$= \lim_{\Delta^k x_0 \to 0} \frac{1}{\Delta^k x_0} \int_{\partial \Delta^k x_0} d^{k-1}x [F(x_0) + (x - x_0) \cdot \nabla_{x_0} F(x_0) + |x - x_0| E(x, x - x_0)]$$

$$= (-1)^{k+1} \lim_{\Delta^k x_0 \to 0} \frac{1}{\Delta^k x_0} \Delta^k x_0 \nabla_{x_0} F(x_0) = (-1)^{k+1} \nabla_{x_0} F(x_0),$$

from which the theorem follows.

Q.E.D.

Theorem 5 *BOUNDARY THEOREM*

$$\int_S G(x)d^k x\partial F(x) = (-1)^{k+1}\int_{\partial S} G(x)d^{k-1}xF(x)$$

.

PROOF:

$$\int_S Gd^k x\partial F = \lim_{N\to\infty}\sum_{i=1}^{N} G(x_i)\Delta^k x_i \partial_{x_i} F(x_i)$$

$$= \lim_{N\to\infty}\sum_{i=1}^{N}\int_{\partial\Delta^k x_i} g(x)\Delta^k x_i \frac{(-1)^{k+1}}{\Delta^k x_i} f(x)$$

$$= (-1)^{k+1}\int_{\partial S} G(x)d^{k-1}xF(x).$$

Q.E.D.

Classical Integration Theorems

The *boundary theorem* for $k = 1, 2, 3$.

For $k = 1$,

$$\int_a^b dx\partial F(x) = \int_{\{b\}-\{a\}} d^0 x f(x) = F(b) - F(a).$$

The left hand side gives

$$\int_a^b dx\partial F(x) = \int_a^b dx\cdot\partial F(x) = \int_a^b dF(x).$$

This is the Fundamental Theorem of Calculus when a and b are points on the x-axis.

For $k = 2$,

$$\int_{S_2} d^2 x\partial F(x) = -\int_{\partial S_2} dxF(x).$$

If $F(x)$ is a vector field, we get *Stokes' Theorem*,

$$\int_{S_2} d^2x\partial \wedge F(x) = -\int_{\partial S_2} dx \cdot F(x),$$

or for $i = e_1e_2e_3$,

$$\int_{S_2} d^2xi\partial \times F(x) = -\int_{\partial S_2} dx \cdot F(x),$$

which reduces to

$$\int_{S_2} |d^2x| n \cdot (\nabla \times F(x)) = \int_{\partial S_2} dx \cdot F(x),$$

for the right hand normal vector n.

For $S_3 \subset I\!R^3$, we have $\partial_x = \nabla_x$, and

$$\int_{S_3} d^3x\nabla F(x) = -\int_{\partial S_3} d^2xF(x). \tag{3.13}$$

Multiplying both sides by $-i = -e_1e_2e_3$ gives

$$\int_{S_3} |d^3x|\nabla F(x) = -\int_{\partial S_3} -id^2xF(x) = \int_{\partial S_3} |d^2x|nF(x).$$

Let $F(x)$ be a vector field. Taking scalar parts of (3.13) gives the *Divergence Theorem:*

$$\int_{S_3} |d^3x|\nabla \cdot F(x) = \int_{\partial S_3} |d^2x|n \cdot F(x).$$

Residue Theorem

Let S be a smooth compact *flat* m-surface lying in E^m. Let

$$g(x, x') = \frac{1}{\Omega}\frac{x - x'}{|x - x'|^m}$$

be the *Green's function* for E^m, where $\Omega = \frac{2\pi^{m/2}}{\Gamma(m/2)}$. The Green's function satisfies

- $\nabla g = -g\nabla'^{\dagger} = \delta(x - x')$, where δ is the famous *delta function distribution.*

- $\int_S |dx'^m|\delta(x-x')F(x') = F(x)$, for a continuous function $F(x)$ at $x \in S$.

Applying the boundary theorem, we get

$$F(x) = -\frac{(-1)^m}{I(x)}\left[\int_S g(x,x')d^m x'\nabla' F(x')\right.$$

$$\left. - \int_{\partial S} g(x,x')d^{m-1}x' F(x')\right]$$

for $x \in S$, and where $I(x)$ is the unit pseudoscalar element of S at x.

Generalized Cauchy's Theorem: If $\nabla F(x) = 0$ in S, then

$$F(x) = \frac{(-1)^m}{I(x)}\int_{\partial S} g(x,x')d^{m-1}x' F(x'),$$

showing that an *analytic* function is determined by its values on the boundary ∂S of S.

Clifford analysis has been highly developed along the lines of functional analysis in [**?**].

Riemannian Geometry

Let S be an *abstract Riemannian k-manifold*, a point $x \in S$ coordinatized by $x = x(x^1, x^2, \ldots, x^k)$. Since there are problems with the vector manifold approach as developed in [**?**] regarding embedding theorems, we wish to give here definitions which do not require an embedding but in which the power of geometric algebra can still be effectively utilized in the study of manifolds.

We wish the gradient operator ∇_x at the point $x \in I\!R^k$, to become the *intrinsic vector derivative* when generalized to the abstract k-manifold S; for (embedded) vector manifolds we use the vector derivative ∂_x discussed earlier. To accomplish this, we assume that a Riemannian k-manifold comes equipped with a *Riemannian vector derivative* or *Riemannian vector connection* ∇_x characterized by the following three properties:

- $\nabla_x x^i = e^i(x)$ for $i = 1, 2, \ldots, k$ where the *vectors* $e^i(x)$ generate the *tangent geometric algebras* G_x at each point $x \in S$.

  The reciprocal basis $\{e_i\}$ to $\{e^i\}$ define the *tangent coordinate vectors* to the respective coordinate curves at the point $x = x(x^1, x^2, \ldots, x^k)$.

- If $a \in G_x$ is a vector, then $a \cdot \nabla_x\ x = a$.

  By a *constant vector* $a = \sum a^i e_i$ we mean a vector whose *components* $a^1, a^2, \ldots, a^k \in I\!R$ are not a function of $x \in S$, *i.e.,* for each i, $\nabla_x a_i = 0$.

- For constant vectors a and b,

  $$a \cdot \nabla_x b = \Gamma_x(a, b),$$

  where $\Gamma_x(a, b)$ is a bilinear vector-valued function in G_x, called the *Christoffel function* at x.

By the *tangent bundle* to the manifold S, we mean

$$G = \cup_{x \in S}\{G_x\}, \tag{3.14}$$

and we call $\nabla = \cup_{x \in S}\{\nabla_x\}$ the *intrinsic vector connection* on the tangent bundle G. The *projection* P_x is then naturally defined by $P_x G = G_x$ and $P_x \nabla = \nabla_x$. The *unit pseudoscalar field* $I(x)$ to S at x is defined by

$$I(x) = e_1 \wedge e_2 \wedge \ldots \wedge e_m / |e_1 \wedge e_2 \wedge \ldots \wedge e_m|.$$

In the special case when $S \subset I\!R^n$, the tangent bundle G becomes a subset of the geometric algebra $I\!R_n \equiv G(I\!R^n)$ of $I\!R^n$, and the projection $P_x(A) = I^{-1}(I \cdot A)$ is the projection onto the sub-geometric algebra G_x at $x \in S$. The vector derivative ∂_x of S can then be written $\partial_x = P_x(\nabla_x)$ where ∇_x is the gradient of $I\!R^n$. Of course, there are many details to consider to guarantee that our theory is compatible with the standard approaches to manifold theory [?].

The main advantage of working in the embedded *vector manifold* is that it is possible to define the bivector-valued *shape operator*

$$S(a) = I^{-1}(a \cdot \partial_x I) = I^{-1} P_a(I)$$

using the vector derivative ∂_x. Slightly more general, for a multivector $A \in G_x$ we can define

$$S(A) \equiv \partial_x P_x(A).$$

We also have the basic relationship

$$\partial_x A = \nabla_x A + S(A)$$

for any multivector field $A(x) \in G$.

Riemannian Curvature is defined by

$$R(a \wedge b) \cdot c \equiv [a \cdot \nabla, b \cdot \nabla]c - [a, b] \cdot \nabla c = (a \wedge b) \cdot (\nabla \wedge \nabla)c.$$

In terms of the shape operator,

$$R(a \wedge b) = P[S_a \times S_b] = P_a(S_b),$$

so curvature is completely determined by $S_a = S(a)$.

The curvature operator is *symmetric*

$$R(A) \cdot B = A \cdot R(B),$$

and satisfies the *Ricci identity*

$$a \cdot R(b \wedge c) + b \cdot R(c \wedge a) + c \cdot R(a \wedge b) = 0.$$

These two conditions are equivalent to

$$\partial_a \wedge R(a \wedge b) = 0.$$

The famous *Bianchi identity* takes the form

$$\nabla_x \wedge R_x(a \wedge b) = 0.$$

In 4-dimensional spacetime the curvature operator is a symmetric endomorphism in the 6-dimensional space of bivectors with the ultra hyperbolic signature $\{+\ +\ +\ -\ -\ -\}$. It would be interesting to carry out a complete classification of this operator via it's minimal polynomial. Such a classification has been carried out for the *Projective Weyl Tensor*

$$W(a \wedge b) \cdot c = R(a \wedge b) \cdot c - \frac{1}{m-1}(a \wedge b) \cdot (\partial_a \cdot R(a \wedge c)),$$

called the *Petrov Classification.*

Let $A_r(x)$ be an r-vector field. The *r-form* of $A_r(x)$ is

$$\alpha_r(dX_r) \equiv dX_r \cdot A_r,$$

where $dX_r = dx_1 \wedge dx_2 \wedge \ldots \wedge dx_r$. The Cartan *exterior derivative* $d\alpha_r$ is defined by

$$d\alpha_r = dX_{r+1} \cdot (\nabla \wedge A_r(x)).$$

The famous formula $d^2\alpha_r = 0$ is a simple consequence of the integrability condition $\nabla \wedge \nabla x^i = \nabla \wedge e^i = 0$:

$$d^2\alpha_r = dX_{r+2} \cdot [\nabla \wedge (\nabla \wedge A_r)]$$

$$= dX_{r+2} \cdot [(\nabla \wedge \nabla) \wedge A_r] = 0.$$

Acknowledgement

I want to thank the William Baylis for organizing the most enjoyable Banff Clifford Algebra Summer School and making possible the development of these lectures.

Bibliography

[1] R. L. Bishop, R. J. Crittenden, *Geometry of Manifolds*, Academic Press, New York (1964).

[2] R. Delanghe, F. Sommen and V. Soucek, *Clifford Algebra and Spinor-valued functions: a function theory for the Dirac-operator*, Mathematics and Its Applications 53, Kluwer, Dordrecht (1992).

[3] C. Doran, D. Hestenes, F. Sommen and N. Van Acker, *Lie groups as spin groups*, J. Math. Phys. **34** (8), p. 3642-3669, August 1993.

[4] W. Fulton and Joe Harris, *Representation Theory*, Springer-Verlag 1991.

[5] F.R. Gantmacher, *Matrix Theory*, Vol. 1, Chelsea Publishing Company, New York, 1960.

[6] D. Hestenes and G. Sobczyk, *Clifford Algebra to Geometric Calculus: A Unified Language for Mathematics and Physics*, 2nd edition, Kluwer 1992.

[7] I. R. Porteous, *Topological Geometry, 2nd edition*, Cambridge University Press, 1981.

[8] M. Riesz, *Clifford Numbers and Spinors*, edited by E. Bolinder and P. Lounesto, Kluwer 1993.

[9] G. Sobczyk, Hyperbolic Number Plane, *The College Mathematics Journal*, (to appear) September 1995.

[10] G. Sobczyk, "Clifford Algebra Techniques in Linear Algebra", in CLIFFORD ALGEBRAS AND SPINOR STRUCTURES, editors: Pertti Lounesto and Rafal Ablamowicz, A Volume dedicated to the memory of Albert Crumeyrolle, Mathematics and Its Applications Vol. 321, Kluwer, p.101-110 (1995).

[11] G. Sobczyk, Unipotents, Idempotents, and a Spinor Basis for Matrices, *Advances in Applied Clifford Algebras* [**2**], No. 1 (1992) 53-62.

[12] G. E. Sobczyk, Jordan Form in Clifford Algebras, *Clifford Algebras and their Applications in Mathematical Physics*, Proceedings of the Third International Clifford Algebras Workshop, Edited by Fred Brackx, Richard Delanghe, and Herman Serras, Kluwer, Dordrecht, 1993.

[13] G. E. Sobczyk, Structure of Factor Algebras and Clifford Algebra, *LINEAR ALGEBRA AND ITS APPLICATIONS*, (to appear) 1995.

[14] G. E. Sobczyk, Structure Equation of a Linear Operator, *The College Mathematics Journal*, (to appear).

[15] G. E. Sobczyk, Simplicial Calculus, *Clifford Algebras and their Applications in Mathematical Physics*, Proceedings of the Second International Clifford Algebras Workshop, Kluwer, Dordrecht, 1988.